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Through-the-thickness stress profiles in laminated composite and sandwich structure plates via unified formulation

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This paper presents a discussion on the through-the-thickness stress profiles of thick laminated plates and sandwich structures calculated with a particular generalized unified formulation. It consists on a finite element Unified Formulation (UF) with C-1 continuity of the transverse displacement field implemented in a four node plate element (CGF- Caliri's Generalized Formulation). The current works on unified formulations attempt to simplify and settle the development, use and choice of plate and shell theories. However, even using unified approaches, no generalizations base on specific problems are possible. For instance, the present work demonstrates that a non-linear plate theory may yield worse results when compared to a linear solution. It was found that, for a selected thick sandwich structure, a cubic layerwise theory provided a set of results 27% less accurate than the linear one. Moreover, when the through-the-thickness stress profiles are closely evaluated, the accuracy of the stress profiles are subjected to the solution method and not only the theory. Particularly, the effects of poor mesh resolution and boundary conditions are demonstrated in this work. Nevertheless, CGF is able to predict point-wise accurate results (less than 1% of deviation) even for the for transverse stresses variables.

1. Introduction

Laminated composite and sandwich structural plates and shells comprise a class of structural components, which is applied on a daily basis in engineering. The structural simplification of 3D (three dimensional) structures into 2D (bi-dimensional) ones is very appealing, but not accomplished without overcoming a few hurdles. For example, wind turbine blade engineering is a top-notch application comprising such structures [1–3]. When designing these blades, the proper calculation of the internal stresses of these layered structures, especially those using thick laminates and sandwich structures [4] is crucial to the performance and life of the wind turbine. The simplification mentioned above is basically the removal of the thickness coordinate out of the problem, yielding the so-called plate/shell theories. Depending on how this simplification is performed, a different theory may emerge. There are, according to Carrera [5], three main approaches: continuum base, asymptotic or axiomatic theories. Each of these structural theories brings a different formulation to accurately estimate the displacement and/or stress fields. However, it is not a straightforward task to identify the origin of the lack of accuracy of a particular theory when compared to another one derived from a different approach. That is why the recent use Unified Formulations is very attractive [6]. By choosing one unified solution method, different plate/shell theories can be properly compared and tested, because the source of the variations in the results are controlled by the engineer [5–20]. The authors also recommend their lengthy review on plate and shell theories in Ref. [21]. It demonstrates, with over 100 papers, the actual infinite number of approaches to the plate/shell problem available in the literature.

Thus, this paper presents the stress profiles obtained by the authors using a finite element based on unified formulation tagged as CGF (Caliri's Generalized Formulation) [22,23]. In fact, CGF is based on Carrera's Unified Formulation [5,6] and the Generalized Unified Formulation from Ref. [11]. The novelty of this solution method is the proposal of a unified finite element (4-node plate element) solution considering C-1 continuity of the transverse displacement field to mainly increase the accuracy of the transverse stresses without using mixed formulations. In the previous works

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$$\boxed{\mathrm{L/E} \mid \mathrm{D/M} \mid \mathrm{C} \mid \begin{smallmatrix} \mathbf{N}_{\sigma_i z_i} \\ \mathbf{N}_{u_i z_i} \end{smallmatrix}}$$

Fig. 1. General form of acronyms used with unified formulations.

Table 1
Material properties of case study II.

Ply	E_1	E_2	E_3	G_{12}	G_{13}	G_{23}	υ_{12}	υ_{13}	υ_{23}
Ply	$25E_2$	1 GPa	1 GPa	$0.5E_2$	$0.5E_2$	$0.2E_2$	0.25	0.25	0.25
Core	$0.04E_2$	$0.04E_2$	$0.5E_2$	$0.016E_2$	$0.06E_2$	$0.06E_2$	0.25	0.1	0.1

of the authors [22,23] thin plates were investigated, as well as thick laminated and sandwich structure plates were studied after an improvement of the formulation. To appreciate the performance of CGF, regarding the stress profiles, several laminated composite plates are investigated and compared, considering the combination of different length-to-thickness ratios (a/h) and plate/shell theories. Additionally, two sandwich structures are investigated separately due to their higher material anisotropy and/or higher geometric aspect ratios.

2. The unified formulation

The results obtained by CGF can be different from those obtained by Carrera and Demasi [5–8,10,11] (named CUF - Carrera's Unified Formulation and GUF - Generalized Unified Formulation) because the proposed finite element solution preserves the C-1 continuity of the transverse displacement field $w(x,y,z)$ using Hermite polynomials. Unified formulations possess a kernel matrix from which the numerous theories stem. In CGF's kernel matrix, the finite element formulation yields in-plane rotations and a twist, which appear in the C-1 formulation process. The elementary equilibrium system for a steady-state case using CGF can be seen in equation (1).

$$\begin{pmatrix} \mathbf{P} \\ \mathbf{R} \end{pmatrix}_i^{(e)} = \begin{bmatrix} \overline{\mathbf{K}}_{uu} & \overline{\mathbf{K}}_{uv} \\ sym & \overline{\mathbf{K}}_{vv} \end{bmatrix}_{ij}^{(e)} \begin{pmatrix} \mathbf{u} \\ \mathbf{w}_{,} \end{pmatrix}_j^{(e)} + \begin{bmatrix} \mathbf{M}_{uu} & 0 \\ sym & \mathbf{M}_{vv} \end{bmatrix}_{ij}^{(e)} \begin{pmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{w}}_{,} \end{pmatrix}_j^{(e)} \tag{1}$$

Respectively, the upper case **P**, **R** and **M** variables stand for normal and moment loads and the mass. The variable $\overline{\mathbf{K}}$ represents the matrix stiffness components. The sub-indices u and w represent the displacement fields and the curvatures, respectively, while the i and j are recursive indexes.

The stiffness matrices in equation (1) are not in their unified form yet and neither of the unification or generalization parameters can be seen in this equation. To obtain an explicit unified kernel matrix, an axiomatic unification of the displacement fields needs to be equated into equation (1).

Carrera's and Demasi's kernels stem from the respective unifications in equations (2) and (3). In these equations, the displacement fields are postulated. The superscript k refers to a particular ply of the layered structure in cases where the formulation is of the layerwise type [6,11].

Table 2
Material properties of case study III.

Ply	E (MPa)	v
Top skin	8000	0.34
Core	1000; 0.1	
Bottom skin	10000	

$$\begin{aligned} u^k(x,y,z) &= F_\tau^k(z)u_\tau^k(x,y) \\ \tau &= 0\ldots N_\tau \end{aligned} \tag{2}$$

The $F_\tau(z)$ variables are the transverse functions; the $u_\tau(x,y)$ are the in-plane functions. The subindex τ is a recursive index for the current order of each component of the complete polynomial of order N_τ assumed for the displacement variable u.

$$\begin{aligned} u^k(x,y,z) &= F_{\alpha_u}^k(z)u_{\alpha_u}^k(x,y); & \alpha_u &= 0\ldots N_{\alpha_u} \\ v^k(x,y,z) &= F_{\alpha_v}^k(z)v_{\alpha_v}^k(x,y); & \alpha_v &= 0\ldots N_{\alpha_v} \\ w^k(x,y,z) &= F_{\alpha_w}^k(z)w_{\alpha_w}^k(x,y); & \alpha_w &= 0\ldots N_{\alpha_w} \end{aligned} \tag{3}$$

In Ref. [11], Demasi generalized the unification process and decoupled the number of terms from each displacement field in equation (2) which produced equation (3). In N_{α_u}, for example, the subscript α has another subscript u to distinguish the order of the polynomials according to each displacement field. In the case of a laminated structure approached as one single monocoque ply (Equivalent Single Layer approach - ESL), the transversal functions $F(z)$ can assume the form of the terms of a Taylor expansion, yielding:

$$\begin{aligned} F_{N_\tau}^k(z) &= F_{N_\tau}(z) \\ F_0(z) &= 1 \\ F_1(z) &= z \\ F_2(z) &= z^2 \\ &\vdots \\ F_{N_\tau}(z) &= z^{N_\tau} \end{aligned} \tag{4}$$

However, if a set of displacement functions is assumed to each ply, then the approach is termed Layerwise (LW) and the transverse functions must satisfy the compatibility requirements of the calculated variables at the interfaces. Carrera and Demasi [5–8,11] pointed to the use of Legendre polynomials to fulfill such task. In Ref. [23] this approach is also detailed. Thus, considering both ESL and LW cases, the finite element kernel of CGF, $\mathbf{K}_{CGF}$, is:

$$\mathbf{K}_{CGF} = \begin{bmatrix} \tilde{\mathbf{K}}_{uu}^{krs} & \tilde{\mathbf{K}}_{uw}^{krs} \\ sym & \tilde{\mathbf{K}}_{ww}^{krs} \end{bmatrix} \tag{5}$$

where

$$\begin{aligned} \tilde{\mathbf{K}}_{uu}^{krs} &= \begin{pmatrix} K_{uu}^{kr_us_u} & K_{uv}^{kr_us_v} & K_{uw}^{kr_us_w} \\ & K_{vv}^{kr_us_v} & K_{vw}^{kr_us_w} \\ sym & & K_{ww}^{kr_us_w} \end{pmatrix} \\ \tilde{\mathbf{K}}_{uw}^{krs} &= \begin{pmatrix} K_{uw_{,x}}^{kr_us_w} & K_{uw_{,y}}^{kr_us_w} & K_{uw_{,xy}}^{kr_us_w} \\ K_{vw_{,x}}^{kr_us_w} & K_{vw_{,y}}^{kr_us_w} & K_{vw_{,xy}}^{kr_us_w} \\ K_{ww_{,x}}^{kr_us_w} & K_{ww_{,y}}^{kr_us_w} & K_{ww_{,xy}}^{kr_us_w} \end{pmatrix} \\ \tilde{\mathbf{K}}_{ww}^{krs} &= \begin{pmatrix} K_{w_{,x}w_{,x}}^{kr_us_w} & K_{w_{,x}w_{,y}}^{kr_us_w} & K_{w_{,x}w_{,xy}}^{kr_us_w} \\ & K_{w_{,y}w_{,y}}^{kr_us_w} & K_{w_{,y}w_{,xy}}^{kr_us_w} \\ sym & & K_{w_{,xy}w_{,xy}}^{kr_us_w} \end{pmatrix} \end{aligned} \tag{6}$$

where the r,s,i and j letters are recursive indexes which come from the derivation of the finite element equilibrium equations based on the Principle of Virtual Displacements (PVD).

As an example, the K_{uw}^{krsij} component is shown in equation (7).