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Note on the stability of viscous roll waves

Note sur la stabilité des roll waves *visqueuses*

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ABSTRACT

In this note, we announce a complete classification of the stability of periodic roll-wave solutions of the viscous shallow water equations, from their onset at Froude number $F \approx 2$ up to the infinite Froude limit. For intermediate Froude numbers, we obtain numerically a particularly simple power-law relation between F and the boundaries of the region of stable periods, which appears potentially useful in hydraulic engineering applications. In the asymptotic regime $F \to 2$ (onset), we provide an analytic expression of the stability boundaries, whereas in the limit $F \to \infty$, we show that roll waves are always unstable.

RÉSUMÉ

Les *roll waves* sont des ondes progressives périodiques hydrodynamiques, modélisées comme des solutions des équations de Saint-Venant. Dans cette note, nous annonçons une classification complète des *roll waves* stables de leur apparition à F (le nombre de Froude) proche de 2 à $F \to \infty$. Pour les nombres de Froude intermédiaires, nous avons mené une étude numérique des critères de stabilité spectrale. Dans le régime asymptotique $F \to 2$, nous donnons une expression analytique des limites de stabilité, alors que pour $F \to \infty$, nous montrons que les *roll waves* sont toujours instables.

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1. Introduction

In this note, we announce the classification in [1–3] of the spectral stability of roll-wave solutions of the "viscous" Saint-Venant equations for an inclined shallow water flow, taking into account drag and viscosity. Written in nondimensional Eulerian form, the shallow water equations for a thin film down an incline are

$$\partial_t h + \partial_x(hu) = 0, \qquad \partial_t(hu) + \partial_x\left(hu^2 + \frac{h^2}{2\,F^2}\right) = h - |u|u + \nu\partial_x(h\partial_x u) \tag{1}$$

where F is the Froude number and $\nu = Re^{-1}$ is the inverse of the Reynolds number. Here $h(x,t)$ denotes the fluid height, whereas $u(x,t)$ is the fluid velocity averaged with respect to height. The terms h and $|u|u$ on the right-hand side of the second equation model, respectively, gravitational force and turbulent friction along the bottom. Roll-waves are well-known hydrodynamic instabilities of (1), arising in the region $F > 2$, for which constant solutions, corresponding to parallel flow, are unstable. They are commonly found in man-made conduits such as aqueducts and spillways, and have been reproduced in laboratory flumes [4]. However, up until now, there has been no complete rigorous stability analysis of viscous Saint-Venant roll waves either at the linear (spectral) or nonlinear level.

Roll-waves may be modeled as periodic wave train solutions of (1). In [2], it was proved for a large class of viscous conservation laws and under suitable spectral assumptions that periodic wave trains are nonlinearly stable (in a spatially modulated sense). In [3,5], this nonlinear analysis has been extended to encompass all periodic wave train solutions of the shallow water system (1) that satisfy those spectral assumptions. The main issue then is the verification of such assumptions. Here, we provide a complete description of the set of stable roll waves of (1): for each Froude number $F > 2$, we exhibit (either theoretically or numerically) the range of spatial periods where stable roll waves are found. To our knowledge, this is the first complete result of stability in the case of shallow water equations. However, let us mention the study in [6]: there, the authors studied the *modulational stability* of Dressler *inviscid* roll waves. A set of modulation equations is derived by assuming that the parameters that encode the roll waves slowly vary in time and space: lack of hyperbolicity of the modulation equations is expected to provide a sufficient criterion for spectral instability of roll waves under special kinds of large-scale perturbations.

In Section 2, we introduce the spectral problem and recall the spectral assumptions that have to be verified in order to obtain nonlinear stability of periodic waves. In Section 3, we consider the intermediate Froude number regime $2 \le F \le 100$. We find a dramatic transition around $F \approx 2.3$ from the small-F description of stability to a remarkably simple power-law description of surfaces bounding from above and below regions in parameter space corresponding to stable waves. These surfaces eventually intersect, yielding instability for all sufficiently large F. In Section 4, we focus on two asymptotic regimes: $F \to 2$ (onset) and $F \to \infty$. As $F \to 2$, the shallow water equations reduce to a generalized Kuramoto–Sivashinsky equation, and we obtain an asymptotic analytic formula for the stability boundaries. As $F \to \infty$, we exhibit a non-trivial regime and an asymptotic model that admits only unstable roll waves, indicating the instability of roll waves for sufficiently large values of F.

2. Formulation of the spectral problem

As the full nonlinear theory is given in Lagrangian coordinates of mass [5,3], for the sake of consistency we rewrite the viscous shallow water system (1) as

$$\partial_t\tau - \partial_x u = 0, \quad \partial_t u + \partial_x\left(\frac{\tau^{-2}}{2\,F^2}\right) = 1 - \tau\, u^2 + \nu\partial_x(\tau^{-2}\partial_x u) \tag{2}$$

where $\tau := 1/h$ and x denotes now a Lagrangian marker rather than a physical location $\tilde{x}$, satisfying the relations $\mathrm{d}\tilde{x}/\mathrm{d}t = u(\tilde{x},t)$ and $\mathrm{d}\tilde{x}/\mathrm{d}x = \tau(\tilde{x},t)$. There is a one-to-one correspondence between periodic waves of the Lagrangian and Eulerian forms. It also holds for the spectral problem in its Floquet-by-Floquet description; see [7]. Thus there is no loss of information in choosing to work with the Lagrangian form. Now we introduce the spectral problem. Denote by $(\bar{\tau},\bar{u},\bar{c})$ a particular periodic traveling (roll-wave) solution to (2) of period X. Linearizing (2) about $(\bar{\tau},\bar{u})$ in the co-moving frame $(x-\bar{c}t,t)$ and seeking modes of the form $(\tau,u)(x,t) = e^{\lambda t}(\tau,u)(x)$, one obtains

$$\begin{aligned}
&(u+\bar{c}\tau)' = \lambda\,\tau \\
&\nu(\bar{\tau}^{-2}u')' = (\lambda + 2\bar{u}\bar{\tau})u - \left(\left(\frac{\bar{\tau}^{-3}}{F^2} - 2\bar{\tau}^{-3}\bar{u}'\right)\tau' + \bar{c}u'\right) + \left(\bar{u}^2 - \left(\frac{\bar{\tau}^{-3}}{F^2} - 2\bar{\tau}^{-3}\bar{u}'\right)'\right)\tau
\end{aligned} \tag{3}$$

where primes denote differentiation with respect to x. Setting $v = (\tau,u)^{\mathrm{T}}$, the spectral problem (3) may be written as $Lv = \lambda v$, where L is a differential operator with *periodic coefficients*. By Floquet's theory, one has that $\lambda \in \sigma_{L^2(\mathbb{R})}(L)$ (the spectrum of L acting on $L^2(\mathbb{R})$) if and only if there are $\xi \in [-\pi/X, \pi/X)$ and $w \in L^2_{\mathrm{per}}([0,X])$ (a function of period X) such that $L_\xi\, w = \lambda w$, where L_ξ is the corresponding Bloch operator defined via $\left(L_\xi\, w\right)(x) := \mathrm{e}^{-\mathrm{i}\xi x}L\left[\mathrm{e}^{\mathrm{i}\xi\cdot}w(\cdot)\right](x)$. Consequently, the spectrum may be decomposed into countably many curves $\lambda(\xi)$ of $L^2_{\mathrm{per}}([0,X])$-eigenvalues of the operators L_ξ. Roll waves are proved to be nonlinearly stable under the following *diffusive spectral stability conditions*: