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Mathematical justification of an elastic elliptic membrane obstacle problem

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ABSTRACT

Starting from the 3D Signorini problem for a family of elastic elliptic shells, we justify that the obstacle problem of an elastic elliptic membrane is the right approximation posed in a 2D domain, when the thickness tends to zero. Specifically, we provide convergence results in the scaled and de-scaled formulations.

1. Introduction

In the last decade, asymptotic methods have been used to derive and justify contact models for beams and plates and, recently, in [1,2] the authors obtained the first results in the justification of obstacle problems as the two-dimensional limit of unilateral frictionless contact problems for the particular case of shallow shells. Additionally, the rigid foundation/obstacle was assumed to be a plane. More recently, in [3], we developed the formal asymptotic analysis of the problem for general elastic shells in frictionless contact with a rigid foundation, without the previously indicated restrictions. From the work in [3], a classification of different limit problems arose, depending upon the geometry of the middle surface and the region where the Dirichlet condition was placed. This classification is the natural extension of what was found by Ciarlet, Sánchez-Palencia et al. in their works for the case without contact, namely, membranes and flexural shells (see [4] and references therein). This Note aims at justifying rigorously that the obstacle problem of an elastic elliptic membrane is the right two-dimensional approximation of the three-dimensional Signorini problem for a family of elastic elliptic shells, when the thickness tends to zero.

2. The three-dimensional Signorini contact problem for elastic shells: variational formulation in curvilinear coordinates

Let ω be a domain of $\mathbb{R}^2$, with a Lipschitz-continuous boundary $\gamma = \partial\omega$. Let $\boldsymbol{y} = (y_\alpha)$ be a generic point of its closure $\bar{\omega}$ and let ∂_α denote the partial derivative with respect to y_α. Let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha(\boldsymbol{y}) := \partial_\alpha \boldsymbol{\theta}(\boldsymbol{y})$ are linearly independent. These vectors form the covariant basis of the tangent plane to the surface $S := \boldsymbol{\theta}(\bar{\omega})$ at the point $\boldsymbol{\theta}(\boldsymbol{y})$. We can consider the two vectors $\boldsymbol{a}^\alpha(\boldsymbol{y})$ of the same tangent plane defined by the relations

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$\boldsymbol{a}^\alpha(\boldsymbol{y})\cdot\boldsymbol{a}_\beta(\boldsymbol{y})=\delta^\alpha_\beta$, which constitute its contravariant basis. We define $\boldsymbol{a}_3(\boldsymbol{y})=\boldsymbol{a}^3(\boldsymbol{y}):=\frac{\boldsymbol{a}_1(\boldsymbol{y})\wedge\boldsymbol{a}_2(\boldsymbol{y})}{|\boldsymbol{a}_1(\boldsymbol{y})\wedge\boldsymbol{a}_2(\boldsymbol{y})|}$ the unit normal vector to S at the point $\boldsymbol{\theta}(\boldsymbol{y})$, where $\wedge$ denotes the vector product in $\mathbb{R}^3$. We can define the first fundamental form, given as the metric tensor, in covariant or contravariant components, respectively, by $a_{\alpha\beta}:=\boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta$, $a^{\alpha\beta}:=\boldsymbol{a}^\alpha\cdot\boldsymbol{a}^\beta$. Here and in what follows, Greek indices take their values in the set $\{1,2\}$, whereas Latin indices do it in the set $\{1,2,3\}$. The second fundamental form, given as the curvature tensor, in covariant or mixed components, respectively, is given by $b_{\alpha\beta}:=\boldsymbol{a}^3\cdot\partial_\beta\boldsymbol{a}_\alpha$, $b^\beta_\alpha:=a^{\beta\sigma}\cdot b_{\sigma\alpha}$, and the Christoffel symbols of the surface S as $\Gamma^\sigma_{\alpha\beta}:=\boldsymbol{a}^\sigma\cdot\partial_\beta\boldsymbol{a}_\alpha$. The area element along S is $\sqrt{a}\,\mathrm{d}y$ where $a:=\det(a_{\alpha\beta})$.

We define the three-dimensional domain $\Omega^\varepsilon:=\omega\times(-\varepsilon,\varepsilon)$ and its boundary $\Gamma^\varepsilon=\partial\Omega^\varepsilon$. We also define the following parts of the boundary, $\Gamma^\varepsilon_+:=\omega\times\{\varepsilon\}$, $\Gamma^\varepsilon_C:=\omega\times\{-\varepsilon\}$, $\Gamma^\varepsilon_0:=\gamma\times[-\varepsilon,\varepsilon]$. Let $\boldsymbol{x}^\varepsilon=(x^\varepsilon_i)$ be a generic point of $\bar\Omega^\varepsilon$ and let ∂^ε_i denote the partial derivative with respect to x^ε_i. Note that $x^\varepsilon_\alpha=y_\alpha$ and $\partial^\varepsilon_\alpha=\partial_\alpha$. Let $\boldsymbol{\Theta}:\bar\Omega^\varepsilon\to\mathbb{R}^3$ be the mapping defined by

$$\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon):=\boldsymbol{\theta}(\boldsymbol{y})+x^\varepsilon_3\boldsymbol{a}_3(\boldsymbol{y})\ \forall\boldsymbol{x}^\varepsilon=(\boldsymbol{y},x^\varepsilon_3)=(y_1,y_2,x^\varepsilon_3)\in\bar\Omega^\varepsilon \tag{1}$$

In [4, Th. 3.1-1], it is shown that if the injective mapping $\boldsymbol{\theta}:\bar\omega\to\mathbb{R}^3$ is smooth enough, the mapping $\boldsymbol{\Theta}:\bar\Omega^\varepsilon\to\mathbb{R}^3$ is also injective for $0<\varepsilon<\varepsilon_0$ small enough and the vectors $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon):=\partial^\varepsilon_i\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ are linearly independent. Therefore, the three vectors $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon)$ form the covariant basis at the point $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$, and $\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)$, defined by the relations $\boldsymbol{g}^{i,\varepsilon}\cdot\boldsymbol{g}^\varepsilon_j=\delta^i_j$, form the contravariant basis at the point $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$. The covariant and contravariant components of the metric tensor are defined, respectively, as $g^\varepsilon_{ij}:=\boldsymbol{g}^\varepsilon_i\cdot\boldsymbol{g}^\varepsilon_j$, $g^{ij,\varepsilon}:=\boldsymbol{g}^{i,\varepsilon}\cdot\boldsymbol{g}^{j,\varepsilon}$, and Christoffel symbols as $\Gamma^{p,\varepsilon}_{ij}:=\boldsymbol{g}^{p,\varepsilon}\cdot\partial^\varepsilon_i\boldsymbol{g}^\varepsilon_j$. The volume element in the set $\boldsymbol{\Theta}(\bar\Omega^\varepsilon)$ is $\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon$ and the surface element in $\boldsymbol{\Theta}(\Gamma^\varepsilon)$ is $\sqrt{g^\varepsilon}\,\mathrm{d}\Gamma^\varepsilon$, where $g^\varepsilon:=\det(g^\varepsilon_{ij})$. Let $\boldsymbol{n}^\varepsilon(\boldsymbol{x}^\varepsilon)$ denote the unit outward normal vector on $\boldsymbol{x}^\varepsilon\in\Gamma^\varepsilon$ and $\hat{\boldsymbol{n}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)$ the unit outward normal vector on $\hat{\boldsymbol{x}}^\varepsilon=\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)\in\boldsymbol{\Theta}(\Gamma^\varepsilon)$. It is verified that (see, [5, p. 41]) $\hat{\boldsymbol{n}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)=\frac{\mathrm{Cof}(\nabla\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon))\boldsymbol{n}^\varepsilon(\boldsymbol{x}^\varepsilon)}{|\mathrm{Cof}(\nabla\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon))\boldsymbol{n}^\varepsilon(\boldsymbol{x}^\varepsilon)|}$. We are particularly interested in the normal components of vectors on $\boldsymbol{\Theta}(\Gamma^\varepsilon_C)$. Recall that on Γ^ε_C, it is verified that $\boldsymbol{n}^\varepsilon=(0,0,-1)$. Also, note that from (1), we deduce that $\boldsymbol{g}^\varepsilon_3=\boldsymbol{g}^{3,\varepsilon}=\boldsymbol{a}_3$, and therefore $g^{33,\varepsilon}=|\boldsymbol{g}^{3,\varepsilon}|=1$. These arguments imply that, in particular, $\hat{\boldsymbol{n}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)=-\boldsymbol{g}_3(\boldsymbol{x}^\varepsilon)=-\boldsymbol{a}_3(\boldsymbol{y})$, where $\hat{\boldsymbol{x}}^\varepsilon=\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ and $\boldsymbol{x}^\varepsilon=(\boldsymbol{y},-\varepsilon)\in\Gamma^\varepsilon_C$. Now, for a field $\hat{\boldsymbol{v}}^\varepsilon$ defined in $\boldsymbol{\Theta}(\bar\Omega^\varepsilon)$, where the Cartesian basis is denoted by $\{\hat{\boldsymbol{e}}^i\}^3_{i=1}$, we define its covariant curvilinear coordinates (v^ε_i) in $\bar\Omega^\varepsilon$ as $\hat{\boldsymbol{v}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)=\hat v^\varepsilon_i(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}^i=:v^\varepsilon_i(\boldsymbol{x}^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)$ with $\hat{\boldsymbol{x}}^\varepsilon=\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$. Therefore, on Γ^ε_C, we have:

$$\hat v_n:=\hat{\boldsymbol{v}}^\varepsilon\cdot\hat{\boldsymbol{n}}^\varepsilon=(\hat v^\varepsilon_i\hat n^{i,\varepsilon})=(\hat v^\varepsilon_i\hat{\boldsymbol{e}}^i)\cdot(-\boldsymbol{g}_3)=(v^\varepsilon_i\boldsymbol{g}^{i,\varepsilon})\cdot(-\boldsymbol{g}_3)=-v^\varepsilon_3$$

Also, since $v^\varepsilon_i n^{i,\varepsilon}=-v^\varepsilon_3$ on Γ^ε_C, it is verified in particular that $\hat v_n=(\hat v^\varepsilon_i\hat n^{i,\varepsilon})=v^\varepsilon_i n^{i,\varepsilon}=-v^\varepsilon_3$.

We assume that $\boldsymbol{\Theta}(\bar\Omega^\varepsilon)$ is a natural state of a shell made of an elastic material, which is homogeneous and isotropic, so that the material is characterized by its Lamé coefficients $\lambda\geq 0$, $\mu>0$. We assume that these constants are independent of ε. We also assume that the shell is subjected to a boundary condition of place; in particular, the displacements field vanishes on $\boldsymbol{\Theta}(\Gamma^\varepsilon_0)$, this is, the whole lateral face of the shell. Further, under the effect of applied volume forces of density $\hat{\boldsymbol{f}}^\varepsilon=(\hat f^{i,\varepsilon})$ acting in $\boldsymbol{\Theta}(\Omega^\varepsilon)$ and tractions of density $\hat{\boldsymbol{h}}^\varepsilon=(\hat h^{i,\varepsilon})$ acting upon $\boldsymbol{\Theta}(\Gamma^\varepsilon_+)$, the elastic shell is deformed and may enter in contact with a rigid foundation, which, initially, is at a known distance s^ε measured along the direction of $\hat{\boldsymbol{n}}^\varepsilon$ on $\boldsymbol{\Theta}(\Gamma^\varepsilon_C)$. For simplicity, we take $s^\varepsilon=0$ in the following.

We deduce that the unilateral contact condition $\hat v_n\leq 0$ in the well-known definition of the set of admissible displacements in Cartesian coordinates is equivalent to $v^\varepsilon_3\geq 0$ in curvilinear coordinates. Therefore, let us define the set of admissible unknowns as follows:

$$K(\Omega^\varepsilon)=\{\boldsymbol{v}^\varepsilon=(v^\varepsilon_i)\in V(\Omega^\varepsilon);\ v^\varepsilon_3\geq 0\text{ on }\Gamma^\varepsilon_C\}$$

where $V(\Omega^\varepsilon)=\{\boldsymbol{v}^\varepsilon=(v^\varepsilon_i)\in[H^1(\Omega^\varepsilon)]^3;\ \boldsymbol{v}^\varepsilon=\boldsymbol{0}\text{ on }\Gamma^\varepsilon_0\}$ is a real Hilbert space with the induced inner product of $[H^1(\Omega^\varepsilon)]^3$. The corresponding norm is denoted by $||\cdot||_{1,\Omega^\varepsilon}$. Note that $K(\Omega^\varepsilon)$ is a non-empty, closed and convex subset of $V(\Omega^\varepsilon)$. We now give in contravariant components the volume forces $f^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon)$, and tractions $h^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon)\sqrt{g^\varepsilon(\boldsymbol{x}^\varepsilon)}\mathrm{d}\Gamma^\varepsilon$. With these definitions, it is straightforward to derive the variational formulation of the Signorini problem in curvilinear coordinates:

Problem 2.1. Find $\boldsymbol{u}^\varepsilon=(u^\varepsilon_i):\Omega^\varepsilon\to\mathbb{R}^3$ such that

$$\begin{aligned}&\boldsymbol{u}^\varepsilon\in K(\Omega^\varepsilon),\quad\int_{\Omega^\varepsilon}A^{ijkl,\varepsilon}e^\varepsilon_{k||l}(\boldsymbol{u}^\varepsilon)(e^\varepsilon_{i||j}(\boldsymbol{v}^\varepsilon)-e^\varepsilon_{i||j}(\boldsymbol{u}^\varepsilon))\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon\\&\quad\geq\int_{\Omega^\varepsilon}f^{i,\varepsilon}(v^\varepsilon_i-u^\varepsilon_i)\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon+\int_{\Gamma^\varepsilon_+}h^{i,\varepsilon}(v^\varepsilon_i-u^\varepsilon_i)\sqrt{g^\varepsilon}\,\mathrm{d}\Gamma^\varepsilon\quad\forall\boldsymbol{v}^\varepsilon\in K(\Omega^\varepsilon)\end{aligned}$$