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A high-order corrector estimate for a semi-linear elliptic system in perforated domains

Vo Anh Khoa

Mathematics and Computer Science Division, Gran Sasso Science Institute, L'Aquila, Italy

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ABSTRACT

We derive in this Note a high-order corrector estimate for the homogenization of a microscopic semi-linear elliptic system posed in perforated domains. The major challenges are the presence of nonlinear volume and surface reaction rates. This type of correctors justifies mathematically the convergence rate of formal asymptotic expansions for the two-scale homogenization settings. As the main tool, we use energy-like estimates to investigate the error estimate between the micro and macro concentrations and between the corresponding micro- and macro-concentration gradients. This work aims at generalizing the results reported in [1,2].

1. Introduction and problem setting

This Note is a follow-up of [2], in which the derivation of a high-order corrector for a microscopic semi-linear elliptic system posed in heterogeneous/perforated domains is concentrated. In the terminology of homogenization, a *corrector* or *corrector estimate* wants to quantify the error between the approximate solution (governed by a certain asymptotic procedure) and the exact solution. Typically, this kind of estimate is helpful also in controlling the approximation error of numerical methods for multiscale problems (e.g., [3,4]). The main result of this Note is Theorem 3.1, where we report the upper bound of the corrector up to an arbitrary high order.

We consider the semi-linear elliptic boundary value problem

$$\mathcal{A}^\varepsilon u_i^\varepsilon \equiv \nabla \cdot \left(-d_i^\varepsilon \nabla u_i^\varepsilon\right) = R_i\left(u_1^\varepsilon, ..., u_N^\varepsilon\right) \quad \text{in } \Omega^\varepsilon$$

associated with the boundary conditions

$$d_i^\varepsilon \nabla u_i^\varepsilon \cdot \mathrm{n} = \varepsilon\left(a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon F_i\left(u_i^\varepsilon\right)\right) \quad \text{across } \Gamma^\varepsilon$$
$$u_i^\varepsilon = 0 \quad \text{across } \Gamma^{ext}$$

for $i \in \{1, ..., N\}$, with $N \geq 2$ being the number of involved concentrations. For simplicity, we refer to this problem as $\left(P^\varepsilon\right)$.

This problem is connected to the Smoluchowski–Soret–Dufour modeling of the evolution of temperature and colloid concentrations [5,6]. Here, $u^\varepsilon := \left(u_1^\varepsilon, ..., u_N^\varepsilon\right)$ denotes the vector of the concentrations, d_i^ε represents the molecular diffusion with R_i being the volume reaction rate and a_i^ε, b_i^ε are deposition coefficients, whilst F_i indicates a surface chemical reaction for the immobile species. Notice that the quantity ε is called the homogenization parameter or the scale factor. Denote by

E-mail addresses: khoa.vo@gssi.infn.it, vakhoa.hcmus@gmail.com.

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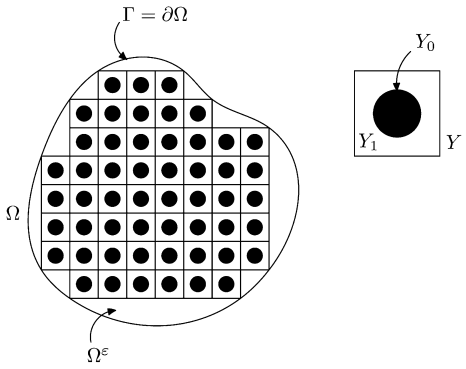

Fig. 1.1. Admissible 2-D perforated domain (left) and basic geometry of the microstructure (right). (By courtesy of Mai Thanh Nhat Truong, Hankyong National University, Republic of Korea.)

$x \in \Omega^\varepsilon$ the macroscopic variable and by $y = x/\varepsilon$ the microscopic variable representing high oscillations at the microscopic geometry. Henceforward, we understand throughout this paper the following convention:

$$d_i^\varepsilon(x) = d_i\left(\frac{x}{\varepsilon}\right) = d_i(y), \quad x \in \Omega^\varepsilon, y \in Y_1$$

with the same meaning for all the oscillating data such as a_i^ε, b_i^ε, etc.

The perforated domain $\Omega^\varepsilon \subset \mathbb{R}^d$ is thought to approximate a porous medium and its precise description can be found in [1,7,2]. As an example, we depict in Fig. 1.1 an admissible geometry of our medium and the corresponding microstructure.

Our corrector estimate evaluation starts from the two-scale asymptotic expansion up to Mth-level ($M \geq 2$) given by

$$u_i^\varepsilon(x) = \sum_{m=0}^{M} \varepsilon^m u_{i,m}(x, y) + \mathcal{O}\left(\varepsilon^{M+1}\right), \quad x \in \Omega^\varepsilon \tag{1}$$

where $u_{i,m}(x, \cdot)$ is Y-periodic for $0 \leq m \leq M$ and $i \in \{1, ..., N\}$.

It is worth noting that in [2], we have analyzed the solvability of (P^ε) using the energy minimization approach and derived the upscaled equations as well as the corresponding effective coefficients. Furthermore, we showed that using the separation of variables, the functions $u_{i,m}(x, y)$ for $0 \leq m \leq M$ can be structured as, e.g.,

$$u_{i,0}(x, y) = \tilde{u}_{i,0}(x)$$
$$u_{i,1}(x, y) = -\chi_{i,1}(y) \cdot \nabla_x \tilde{u}_{i,0}(x)$$
$$u_{i,2}(x, y) = \chi_{i,2}(y) \nabla_x^2 \tilde{u}_{i,0}(x)$$

with $\tilde{u}_{i,0}(x)$ being determined uniquely from the auxiliary problem and $\chi_{i,m}$ satisfying the corresponding cell problems. One can also rule out the $\tilde{u}_{i,0}$-based construction of $u_{i,m}$ that $u_{i,m}(x, y) = (-1)^m \chi_{i,m}(y) \nabla_x^m \tilde{u}_{i,0}(x)$ for $1 \leq m \leq M$.

In this scenario, we wish to obtain the error estimate up to a high-order expansion for the differences of concentrations and their gradients, albeit some types have been investigated so far. In particular, we prove in this Note a corrector in the form of

$$u^\varepsilon - \sum_{k=0}^{K} \varepsilon^k u_k - m^\varepsilon \sum_{m=K+1}^{M} \varepsilon^m u_m \tag{2}$$

in which we fix $K \in \mathbb{N}$ such that $0 \leq K \leq M-2$ and $m^\varepsilon \in C_c^\infty(\Omega)$ is a cut-off function such that $\varepsilon \left|\nabla m^\varepsilon\right| \leq C$ and

$$m^\varepsilon(x) := \begin{cases} 1, & \text{if } \operatorname{dist}(x, \Gamma) \leq \varepsilon \\ 0, & \text{if } \operatorname{dist}(x, \Gamma) \geq 2\varepsilon \end{cases}$$

(see [1] for more properties of m_ε).

With the above definition of m^ε, the second term in (2) vanishes everywhere except in a neighborhood of the boundary of Ω^ε. In other words, the appearance of m^ε provides that the speed of convergence in the interior of the material is better than the rate at the vicinity of the boundary, albeit the standard result expected that $\left\|u^\varepsilon - u_0\right\|_{H^1(\Omega^\varepsilon)} \leq C\varepsilon^{1/2}$. It is then easy to see that (2) includes the cases

$$u^\varepsilon - \sum_{m=0}^{M} \varepsilon^m u_m \quad \text{and} \quad u^\varepsilon - u_0 - m^\varepsilon \sum_{m=1}^{M} \varepsilon^m u_m, \quad M \geq 2$$

reported in [2] and further in [1].

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