C. R. Mecanique ••• (••••) •••–•••

Contents lists available at ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

On Hashin–Shtrikman-type bounds for nonlinear conductors

Michaël Peigney

Université Paris-Est, Laboratoire Navier (UMR 8205), CNRS, École des ponts ParisTech, IFSTTAR, 77455 Marne-la-Vallée, France

ARTICLE INFO

Article history:
Received 16 January 2017
Accepted 28 February 2017
Available online xxxx

Keywords:
Composite materials
Nonlinear homogenization
Bounds
Translation method

ABSTRACT

For linear composite conductors, it is known that the celebrated Hashin–Shtrikman bounds can be recovered by the translation method. We investigate whether the same conclusion extends to nonlinear composites in two dimensions. To that purpose, we consider two-phase composites with perfectly conducting inclusions. In that case, explicit expressions of the various bounds considered can be obtained. The bounds provided by the translation method are compared with the nonlinear Hashin–Shtrikman-type bounds delivered by the Talbot–Willis (1985) [2] and the Ponte Castañeda (1991) [3] procedures.

1. Introduction

In a seminal work [1], Hashin and Shtrikman have obtained optimal bounds on the effective conductivity of linear composite conductors with statistically isotropic microstructures. Those bounds are explicit functions of the volume fractions and conductivities of each constitutive phase. Several methods have been proposed to extend the results of Hashin and Shtrikman to nonlinear composites. A first method, proposed by Talbot and Willis [2], makes uses of a *homogeneous* linear comparison medium and generalizes the variational approach introduced by Hashin and Shtrikman. A second method, due to Ponte Castañeda [3], employs a *heterogeneous* linear comparison medium, i.e. a linear comparison composite. Using that last method, any bound on the effective conductivity of the linear comparison composite can be used to generate a corresponding bound for the nonlinear composite. In particular, when the linear Hashin–Shtrikman bound is used, nonlinear Hashin–Shtrikman-type bounds are obtained. A third method, known has the translation method [4], has been introduced independently by Lurie and Cherkaev [5], Murat and Tartar [6]. Originally introduced in the linear context, that method has proved to be very fruitful in a lot of nonlinear homogenization problems [7–12]. For nonlinear isotropic conductors, the translation method has been used to obtain explicit bounds for composites governed by threshold-type energy functions [13,14].

The three methods mentioned above can generate nonlinear bounds of the Hashin–Shtrikman type, i.e. bounds that hold for the whole class of isotropic composites with prescribed volume fractions and conduction properties of the individual phases. As those methods are not mathematically equivalent, it is important to understand the relations between them. The relation between the Talbot–Willis and the Ponte Castañeda methods has been investigated in [15]: for two-phase composites with perfectly conducting inclusions, the two methods have been proved to give the same results if the energy function of the matrix satisfies a certain strong convexity condition. If that condition is violated, the Talbot–Willis method leads to stronger bounds than the Ponte Castañeda method. The relations with the translation method have been studied in [13]: Bounds obtained from the translation method have been proved to be always at least as good as those provided by

E-mail address: michael.peigney@polytechnique.org.

http://dx.doi.org/10.1016/j.crme.2017.02.006

the Ponte Castañeda method. For two-dimensional composites governed by threshold-type energy functions, the translation method actually gives the same bounds as the Ponte Castañeda method. Numerical results suggest that the same conclusion extends to power-law composites, although a rigorous proof is still lacking.

The objective of this paper is to fill some of the gaps in the relations between the translation method and the methods of Ponte Castañeda, Talbot and Willis by investigating conditions under which the translation method may bring a genuine improvement. To that purpose, we consider two-phase composites with perfectly conducting inclusions, in two dimensions. The formulation of the translation method in that context is presented in Sect. 2. If the energy function of the matrix satisfies the strong convexity assumption introduced in [15], we prove in Sect. 3 that the translation method gives the same bounds as the Talbot–Willis and Ponte Castañeda methods. This confirms the numerical observations made in [13] for the special case of power-law composites. For the translation method to bring a genuine improvement, it is therefore necessary that the energy in the matrix is not strongly convex. Building on that observation, in Sect. 4 we provide an example for which bounds obtained from the translation method are indeed stronger than the Talbot–Willis and Ponte Castañeda bounds.

2. Bounding the effective energy via the translation method

Consider a two-dimensional inhomogeneous electric conductor occupying a domain Ω of unit volume. The electric field $\boldsymbol{e}$ and the current density $\boldsymbol{j}$ are related by the local constitutive law

$$\boldsymbol{j} = \frac{\partial w}{\partial \boldsymbol{e}}(\boldsymbol{e}, \boldsymbol{x}) \tag{1}$$

where the convex energy-density function w depends on the location $\boldsymbol{x}$. Denoting by $\bar{\boldsymbol{e}}$ (resp. $\bar{\boldsymbol{j}}$) the spatial average of $\boldsymbol{e}$ (resp. $\boldsymbol{j}$), the effective constitutive law reads as [16,17]

$$\bar{\boldsymbol{j}} = \frac{\mathrm{d}w_{\text{eff}}}{\mathrm{d}\bar{\boldsymbol{e}}}(\bar{\boldsymbol{e}}) \tag{2}$$

where w_{eff} is the effective energy function of the composite material, defined by

$$w_{\text{eff}}(\bar{\boldsymbol{e}}) = \inf_{\boldsymbol{e} \in K(\bar{\boldsymbol{e}})} \int_{\Omega} w(\boldsymbol{e}, \boldsymbol{x})\, \mathrm{d}\omega \tag{3}$$

In (3), $K(\bar{\boldsymbol{e}})$ is the set of admissible electric fields, as defined by

$$K(\bar{\boldsymbol{e}}) = \{\boldsymbol{e} : \Omega \mapsto \mathbb{R}^2 | \boldsymbol{e} = \nabla V \text{ for some } V : \Omega \mapsto \mathbb{R} \text{ verifying } V(\boldsymbol{x}) = \bar{\boldsymbol{e}} \cdot \boldsymbol{x} \text{ on } \partial\Omega\}$$

Following [5,6], a lower bound on w_{eff} can be obtained by embedding the original problem in a problem of dimension 4. In more detail, we introduce extended fields $\boldsymbol{E}(\boldsymbol{x}) = (\boldsymbol{e}_1(\boldsymbol{x}), \boldsymbol{e}_2(\boldsymbol{x}))$ obtained by considering 2 electric fields $\boldsymbol{e}_1(\boldsymbol{x})$ and $\boldsymbol{e}_2(\boldsymbol{x})$, written side by side. Introducing the *extended energy*

$$W(\boldsymbol{E}, \boldsymbol{x}) = w(\boldsymbol{e}_1, \boldsymbol{x}) + w(\boldsymbol{e}_2, \boldsymbol{x})$$

as well as the *extended effective energy*

$$W_{\text{eff}}(\bar{\boldsymbol{E}}) = w_{\text{eff}}(\bar{\boldsymbol{e}}_1) + w_{\text{eff}}(\bar{\boldsymbol{e}}_2) \tag{4}$$

it can be readily seen from (3) that

$$W_{\text{eff}}(\bar{\boldsymbol{E}}) = \inf_{\boldsymbol{E} \in \mathcal{K}(\bar{\boldsymbol{E}})} \int_{\Omega} W(\boldsymbol{E}, \boldsymbol{x})\, \mathrm{d}\omega \tag{5}$$

where $\bar{\boldsymbol{E}} = (\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2)$ and $\mathcal{K}(\bar{\boldsymbol{E}}) = \{(\boldsymbol{e}_1, \boldsymbol{e}_2) : \boldsymbol{e}_i \in K(\bar{\boldsymbol{e}}_i)\}$. We now proceed to bound W_{eff} from below. To that purpose, it is convenient to represent extended fields $\boldsymbol{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2)$ by 2×2 matrices, i.e.

$$\boldsymbol{E} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

where u_i and v_i are the components of the electric field $\boldsymbol{e}_i$ in a reference basis of $\mathbb{R}^2$. For any scalar α and any $\boldsymbol{T}$ in $\mathbb{R}^{2\times 2}$, consider the Legendre transform

$$(W - \alpha \det)^*(\boldsymbol{T}, \boldsymbol{x}) = \sup_{\boldsymbol{E}} \boldsymbol{E} : \boldsymbol{T} - W(\boldsymbol{E}, \boldsymbol{x}) + \alpha \det \boldsymbol{E} \tag{6}$$

where $\boldsymbol{E} : \boldsymbol{T} = \operatorname{tr} \boldsymbol{E}\boldsymbol{T}$ and det is the determinant in $\mathbb{R}^2$, i.e. $\det \boldsymbol{E} = u_1 v_2 - u_2 v_1$. For any $\boldsymbol{E} \in \mathcal{K}(\bar{\boldsymbol{E}})$, it follows from (6) that

$$W(\boldsymbol{E}, \boldsymbol{x}) \geq \boldsymbol{E}(\boldsymbol{x}) : \boldsymbol{T} + \alpha \det \boldsymbol{E}(\boldsymbol{x}) - (W - \alpha \det)^*(\boldsymbol{T}, \boldsymbol{x}) \tag{7}$$

Please cite this article in press as: M. Peigney, On Hashin–Shtrikman-type bounds for nonlinear conductors, C. R. Mecanique (2017), http://dx.doi.org/10.1016/j.crme.2017.02.006