C. R. Mecanique 344 (2016) 769–775

Contents lists available at ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

Variational discretizations for the dynamics of fluid-conveying flexible tubes

Discrétisations variationnelles pour l'étude de la dynamique de tubes flexibles avec écoulement interne

François Gay-Balmaz [a,*], Vakhtang Putkaradze [b]

[a] *LMD/IPSL, CNRS, École normale supérieure, PSL Research University, École polytechnique, Université Paris-Saclay, Sorbonne Universités, UPMC Université Paris-6, 24, rue Lhomond, 75005 Paris, France*
[b] *Department of Mathematical and Statistical Sciences, Department of Chemical and Materials Engineering, University of Alberta, Edmonton, AB T6G 2G1 Canada*

ARTICLE INFO

Article history:
Received 23 May 2016
Accepted 17 August 2016
Available online 31 August 2016

Keywords:
Variational integrators
Fluid–structure interaction
Fluid-conveying tubes

Mots-clés :
Intégrateurs variationnels
Interaction fluide–structure
Tubes avec écoulement interne

ABSTRACT

We derive a variational approach for discretizing fluid–structure interactions, with a particular focus on the dynamics of fluid-conveying elastic tubes. Our method is based on a discretization of the fluid's back-to-labels map and a Lie group discretization of the tube's variables, coupled with an appropriately formulated discrete version of the fluid conservation law. This approach allows the development of geometric numerical schemes for the dynamics of fluid-conveying collapsible tubes, which preserve several intrinsic geometric properties of the continuous system, such as symmetries and symplecticity. In addition, our approach can also be used to derive simplified, but geometrically consistent, low-component models for further analytical and numerical analysis of the system.

RÉSUMÉ

Nous proposons une approche variationnelle pour la discrétisation d'interactions fluide–structure, en nous focalisant sur la dynamique de tubes élastiques avec écoulement interne. Notre approche est basée sur une discrétisation des trajectoires inverses du fluide et une discrétisation de type groupe de Lie des variables du tube élastique, couplée à une discrétisation appropriée de la contrainte de préservation du volume de fluide. Notre approche permet le développement de schémas numériques géométriques pour la dynamique des tubes souples avec écoulement interne, qui préservent plusieurs propriétés géométriques intrinsèques du système continu, telles que les symétries et la symplecticité. De plus, notre approche peut être utilisée pour produire des modèles simplifiés et

---

* Corresponding author.
E-mail addresses: francois.gay-balmaz@lmd.ens.fr (F. Gay-Balmaz), putkarad@ualberta.ca (V. Putkaradze).

http://dx.doi.org/10.1016/j.crme.2016.08.004

géométriquement consistants, appropriés pour des études analytiques et numériques plus approfondies de ce système.

---

1. Background of the studies in dynamics of fluid-conveying flexible tubes

In this paper, we consider variational discretizations for systems coming from fluid–structure interactions, with a particular emphasis on the dynamics of fluid-conveying tubes. For such systems, the key to the dynamics lies in the interaction of a moving elastic body with a moving fluid. From a mathematical point of view, the Lagrangian description of these systems involves both left-invariant (elastic) and right-invariant (fluid) quantities. For the case of fluid-conveying tubes, an instability appears when the flow rate through the tube exceeds a certain critical value. While this phenomenon has been known for a very long time, the quantitative research in the field started around the middle of the 20th century [1]. Benjamin [2,3] was perhaps the first to formulate a quantitative theory for the 2D dynamics by considering a linked chain of fluid-conveying tubes and by using an augmented Hamilton principle of critical action that takes into account the momentum of the jet leaving the tube. A continuum equation for the linear disturbances was then derived as the limit of the discrete system. The same linearized equation, considered by Païdoussis and collaborators [4] from force balance considerations, formed the basis for further stability analysis of this problem for finite tubes [5–11]. This theory has shown a reasonable agreement with the onset of the instability observed experimentally [7,12–15]. Nonlinear deflection models were also considered in [11,16–18], and a more detailed 3D theory of motion was developed in [19], based on a modification of the full Cosserat rod dynamics. Previous works have also approached this problem as a paradigm for the *follower force approach*, which treats the system as an elastic beam with a follower force that remains tangent to the end of the tube and models the effect of the jet leaving the nozzle [20]. However, once the length of the tube becomes large, the validity of the follower force approach has been questioned, see [21] for a lively and thorough discussion.

The main drawback of the previous approaches lies in the difficulty to incorporate in a consistent way the change of the cross section in the dynamics, which is known as a challenging problem for the so-called collapsible tube. Previous works based on traditional frameworks have considered the change of cross section only through the quasi-static approximation: if $A(s,t)$ is the local cross-sectional area, and $u(s,t)$ is the local velocity of the fluid, with s being the coordinate along the tube and t the time, then the quasi-static assumption states that $uA = \mathrm{const}$ [11,16,17,22]. Unfortunately, this simple law is not correct in general and should not be used unless there is a guarantee that the velocity equilibrates on a faster time scale than the change of A and of the tube properties. This problem has been addressed in our previous works [23,24], where we have developed a geometrically exact setting for dealing with a variable cross-section, and studied the important effects of cross-sectional changes on both linear and nonlinear dynamics. The nonlinear theory was derived from a variational principle in a rigorous geometric setting and for general Lagrangians. It can incorporate general boundary conditions and arbitrary deviations from equilibrium in the three-dimensional space.

The theory derived in [23,24] raised the question of writing consistent approximations of the solutions, both from the point of view of deriving simplified reduced models and of developing structure preserving numerical schemes. The present paper deriving such consistent approximations can be viewed as the development of ideas put forward by Benjamin [2,3] for the case of nonlinear dynamics in three dimensions and with cross-sectional dependence. We also note that nonlinear equations for deformations were derived as a limit of a linked chain in [8] and subsequently used in [13]; however, this derivation was not variational and does not generalize to the case of cross-sectional change we are interested in. In the limit of small spatial and/or temporal steps, our discrete equations converge to the continuous equations obtained in [23,24]. We will base our method on the recent works [25,26] on the multisymplectic discretization of an elastic beam in $\mathbb{R}^3$, which were in turn based on the geometric variational spacetime discretization of Lagrangian field theories first developed in [27]. We refer to [28] for an extensive review of time-variational integrators in Lagrangian mechanics. There are two major difficulties that we will need to address here, namely, the appropriate coupling of the fluid with the elastic tube, and the treatment of the constraint of fluid volume conservation. As it turns out, both of these considerations lead to interesting new concepts, which have not been addressed in the previous literature.

2. A brief introduction to the variational dynamics of a collapsible fluid-conveying tube in the continuous case

In this section, we briefly review the variational formulation for the dynamics of a fluid-conveying collapsible tube developed in [23,24], as it plays a fundamental role in the present paper. The interested reader may consult these articles for the complete treatment of the variational approach, as well as for detailed discussions on boundary conditions, linearized stability and fully nonlinear solutions.

To describe the tube's dynamics, we use the framework of geometrically exact rod theory [29], which is equivalent to the Kirchhoff rod theory for purely elastic rods. The configuration of the tube deforming in space is defined by: (i) the position of its line of centroids given by the map $(s,t) \mapsto \boldsymbol{r}(s,t) \in \mathbb{R}^3$, and (ii) the orientation of the tube's cross sections at the points $\boldsymbol{r}(s,t)$, defined by using a moving orthonormal basis $\mathbf{d}_i(s,t)$, $i = 1,2,3$. The moving basis is described by an