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Anti-plane crack solutions in higher-order elasticity

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ABSTRACT

The asymptotic solutions for cracks in a linear elastic medium under plane or anti-plane state were obtained by Williams (1957) nearly sixty years ago. However, solutions for cracks in second-order elasticity are unavailable, in contrast to those based on the neo–Hookean models, e.g., Knowles (1997). This paper addresses the formulation and solution of crack problems under finite anti-plane deformation using higher-order elasticity. It is found that: (1) a combined second- and third-order elasticity is necessary to ensure that full equilibrium is satisfied, (2) the equilibrium equations are non-homogeneous partial differential equations (pde's) with variable coefficients, and (3) exact particular solutions can be obtained while the homogeneous pde's reduce to nonlinear eigenvalue problems that can be solved numerically. The results show that: (1) the displacement or stress dependence on the radial coordinate is generally a function of the elastic constants, (2) the Piola–Kirchhoff stress matrix is fully populated, with induced normal stresses, in-plane as well as out-of-plane shear stresses, (3) singular stresses at the crack tip may exist, depending on the eigenvalues, and (4) normal stresses are predicted on the crack faces, which may lead to anomalous mechanical behavior in soft biological materials.

1. Introduction

Many biological materials contain soft elastomers/hydrogels, and may undergo significant nonlinear finite deformation. Examples are muscles, cartilage, skins, tendons and cells. The deformation and stresses in such soft materials are of great interest, e.g., contractile stresses in cells drive shape changes in order to regulate physiological processes (Murrell, Oakes, Lenz, & Gardel, 2015), and swelling with accompanying stress singularities may occur in brain tissues (Goriely, Weickenmeier, & Kuhl, 2016). The tearing and fracture of these materials are also of much concern to the biological functions and survival of the tissues/organisms (Casares et al., 2015). Artificial devices such as soft robots, actuators and machines, and replacement organs and tissues, must also be sufficiently tough to withstand fracture. The fracture mechanics of both natural and synthetic soft matter is thus of considerable interest in medical science and engineering.

Finite deformation and elastic nonlinearity are often characteristic of the mechanical behavior of soft solids. That linear elasticity is inadequate for describing the crack tip behavior in such soft solids has been investigated experimentally. Using a high resolution camera and a passive tracer field imprinted on polyacrylamide gels under tensile load, Livne, Bouchbinder, Svetlizky, and Fineberg (2010) showed that the crack tip profile deviated significantly from linear elastic fracture mechanics (LEFM) prediction, but predictions derived from a weakly nonlinear theory can yield a good match with the profile. Indeed, the authors are of the opinion that elastic nonlinearity must occur in the high strain region at the crack tip, even in a macroscopically brittle material. Similarly, Lefranc and Bouchaud (2014) investigated experimentally the mode I fracture of

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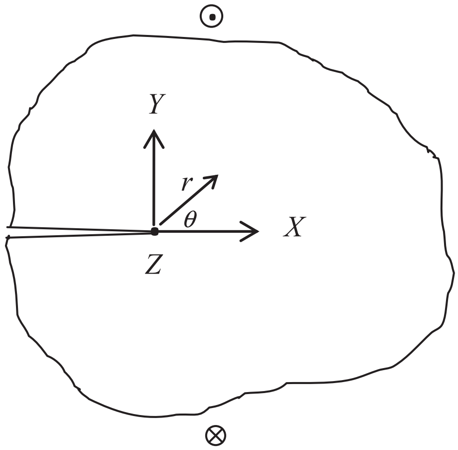

Fig. 1. Schematic of a semi-infinite crack in a nonlinear elastic solid under finite antiplane deformation. The referential coordinate system $X-Y-Z$ is attached to the crack tip. The corresponding referential polar coordinates r and θ are denoted in lower case.

a biopolymer gel, and demonstrated that LEFM breaks down at a certain distance from the crack tip. This distance increases with the crack velocity. The deviation from LEFM was attributed to the presence of large deformation in the gel.

A fundamental problem in fracture mechanics is the determination of the dominant asymptotic stress and displacement field in the vicinity of a crack. This can be regarded as essential towards the basic understanding of fracture, for it permits the development of measures (such as the crack opening displacement and the stress intensity factor) for the quantification of the severity inflicted upon the material due to the crack. This has long been done in the framework of linear elasticity. Williams (1957) investigated the asymptotic fields around a semi-infinite crack in a homogeneous linear elastic solid by solving a biharmonic equation, essentially a combination of the equilibrium and compatibility equations. Subjecting the crack faces to traction-free boundary conditions, an eigenvalue problem permits the determination of the asymptotic stress and displacement, which display respectively an inverse square root and a square root dependence on the radial coordinate with origin at the crack tip.

The determination of such asymptotic fields in a nonlinear solid undergoing finite deformation, however, is non-trivial, due to the complicated equilibrium equations resulting from even a simple energy density function for a nonlinear solid. Analytical solutions are nearly impossible, and numerical solutions cannot be easily obtained. Attempts have been made by Geubelle and Knauss (1994), Knowles (1977), Knowles and Sternberg (1973), Tarantino (1996), Wong and Shield (1969), and more recently by Long, Krishnan, and Hui (2011). In these works, hyperelastic solids described by neo–Hookean work functions are considered. As an example, the work function W used by Knowles (1977) takes the form $W = \mu/2b \times [(1+b(I-3)/n)^n - 1]$, where μ is the shear modulus, b, n are material parameters ($n=1$ yields the neo–Hookean solid), and I is the sum of the square of the principal stretches. Difficulties in obtaining the solutions were met when $n < ½$ or $n > 1.4$. Long et al. (2011) overcame some of these difficulties and showed that solutions could be obtained for any $n > ½$. Nevertheless, the form of the asymptotic solutions is complex, containing functions that are determined by numerically solving second-order differential equations.

In this paper, a semi-infinite crack in an infinite solid under anti-plane finite deformation is studied using higher-order elasticity, in which the strain energy density function is expressed in the form of invariants of the Green-Lagrange tensor. Higher-order continuum elasticity theories have long been considered important in the investigation of crystal anharmocity, when atomic displacements are not small compared to interatomic spacings (Hiki, 1970; Murnaghan, 1951). More recently, they have also been used to study a host of problems, e.g., Acoustoelasticity in soft solids (Gennisson et al., 2007) and soft composites under generalized shear, torsion and axial loading (Wang & Wu, 2014a,b). Solutions for an anti-plane crack within the framework of higher-order elasticity are, to the best of knowledge, presently not available. Indeed, this research shows that several challenges in both the formulation and solution strategy must be overcome before the asymptotic fields can be determined.

Section 2 provides the details of the formulation. Section 3 gives several examples of the displacement and stress solutions. This is followed by a discussion in Section 4, highlighting the significance of the results. Conclusions for the present work are given in Section 5.

2. Problem definition and formulation

2.1. Problem definition

Consider a semi-infinite crack in an infinite solid under finite anti-plane deformation, as shown in Fig. 1. The solid is assumed to be nonlinear, and described within the framework of higher-order elasticity. The elasticity is defined to be