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A multilayered circular inhomogeneity with interfacial diffusion and sliding

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ABSTRACT

We study the time-dependent and plane strain deformations of a circular inhomogeneity bonded to an infinite matrix through a multilayer interphase when subjected to uniform remote stresses. Both interfacial diffusion and rate-dependent sliding occur on all the existing interfaces between two neighboring phases. The problem is solved effectively and elegantly by means of the complex variable method together with the state-space approach. The state variables are just the unknown coefficients appearing in the analytic functions defined in all the phases of the composite. Numerical results are presented to demonstrate the obtained solution.

1. Introduction

Stress relaxation around a circular or spherical inhomogeneity caused by interfacial diffusion and/or rate-dependent sliding has been extensively examined by researchers (see, for example, He & Hu, 2002; He & Zhao, 2000; Koeller & Raj, 1978; Mori, Okabe, & Mura, 1980; Onaka, Huang, Wakashima, & Mori, 1998, 1999; Wang & Pan, 2010; Wang, Wang, & Schiavone, 2016). The long-range mass transport by diffusion is driven by gradients of the normal traction (or gradients of chemical potential) along the interfaces whilst the short-range mass transport by sliding is due to the asperities on the interfaces leading to tension and compression (Herring, 1950; Raj & Ashby, 1971; Sofronis & McMeeking, 1994; Wei, Bower, & Gao, 2008). In the previous discussions, it was assumed that the inhomogeneity is bonded to the surrounding infinite matrix through a sharp interface permitting diffusion and/or sliding. In the modern design of composites, however, an intermediate graded interphase composed of multiple interphase layers with stepwise homogeneous thermoelastic properties exists between the internal inhomogeneity and the surrounding matrix mainly with a goal to reduce thermal stresses (Ru, 1999; Suresh & Mortensen, 1997; Tanaka, Tanaka, Watanabe, Poterasu, & Sugano, 1993). In the solution by Ru (1999) for a circular inhomogeneity with a stepwise graded interphase under thermomechanical loadings, the inhomogeneity, interphase layers and the matrix are perfectly bonded across the concentric circular interfaces.

In this work, we study the transient stress relaxation around a circular elastic inhomogeneity with $N-2$ co-axial interphase layers caused by the combination of diffusion and sliding on all the existing interfaces when the matrix is subjected to uniform in-plane stresses at infinity. The problem is solved by means of the complex variable method (Muskhelishvili, 1953)

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and the state-space approach in which the state variables are just the $4N$ unknown coefficients in the analytic functions defined in the N phases of the composite. The general solution to the state-space equation is derived by solving a generalized eigenvalue problem and by imposing the condition that there is no displacement jump across the interfaces at the initial time. Several numerical examples are then presented to demonstrate the obtained solution.

2. Basic formulation

For plane-strain deformations of elastically isotropic materials, the stresses (σ_{ij}), the associated displacements (u_1, u_2) and the stress functions (ϕ_1, ϕ_2) can be expressed concisely in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2 = r\exp(\mathrm{i}\theta)$ with r and θ being the polar coordinates as (Muskhelishvili, 1953; Ting, 1996)

$$\begin{aligned}&\sigma_{11}+\sigma_{22}=2\left[\varphi'(z)+\overline{\varphi'(z)}\right],\\&\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12}=2\left[\bar{z}\varphi''(z)+\psi'(z)\right],\end{aligned}\tag{1}$$

$$\begin{aligned}&2\mu(u_1+\mathrm{i}u_2)=\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)},\\&\phi_1+\mathrm{i}\phi_2=\mathrm{i}\left[\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}\right],\end{aligned}\tag{2}$$

where $\kappa = 3-4\nu$ with $\nu(0\le\nu\le 1/2)$ being the Poisson's ratio, and μ is the shear modulus. In addition, the stresses are related to the stress functions through (Ting, 1996)

$$\begin{aligned}&\sigma_{11}=-\phi_{1,2},\quad \sigma_{12}=\phi_{1,1},\\&\sigma_{21}=-\phi_{2,2},\quad \sigma_{22}=\phi_{2,1}.\end{aligned}\tag{3}$$

As shown in Fig. 1, we consider the plane-strain deformations of a circular elastic inhomogeneity bonded to an infinite matrix through $N-2$ concentric circular interphase layers. Let S_1, $S_k(k=2,3,\ldots,N-1)$ and S_N denote the inhomogeneity, the $N-2$ interphase layers and the matrix, respectively, which are imperfectly bonded across the $N-1$ concentric circular circles $r=R_k(k=1,2,\ldots,N-1)$. The subscript k or the superscript (k) is used to denote the quantities in S_k. The matrix is subjected to remote uniform in-plane stresses $(\sigma_{11}^{\infty},\sigma_{22}^{\infty})$.

The interfacial diffusion and sliding conditions are specified as follows

$$\begin{aligned}&\sigma_{rr}^{(k+1)}=\sigma_{rr}^{(k)},\quad \sigma_{r\theta}^{(k+1)}=\sigma_{r\theta}^{(k)},\\&\dot{u}_r^{(k)}-\dot{u}_r^{(k+1)}=\frac{D_k}{R_k^2}\frac{\partial^2\sigma_{rr}^{(k)}}{\partial\theta^2},\quad \eta_k\left[\dot{u}_\theta^{(k+1)}-\dot{u}_\theta^{(k)}\right]=\sigma_{r\theta}^{(k)},\quad \text{at } r=R_k,\ k=1,2,\ldots,N-1,\end{aligned}\tag{4}$$

where the overdot denotes differentiation with respect to the time t; D_k and η_k are respectively the interface diffusion constant and viscosity for the interface $r=R_k$.

3. The general solution

When the remote loading is hydrostatic ($\sigma_{11}^{\infty}=\sigma_{22}^{\infty}$), the interfacial diffusion and sliding are both absent on all the interfaces due to the fact that the uniform normal stress and vanishing tangential stress are achieved along each interface. In this case, all the interfaces are perfect and the corresponding solution was obtained by Ru (1999). In this work, we will concentrate on the discussion of a non-hydrostatic loading ($\sigma_{22}^{\infty}=-\sigma_{11}^{\infty}=\sigma_0$). By using a superposition scheme, the solution to a general uniform remote loading containing both the hydrostatic part and the non-hydrostatic part can be conveniently arrived at. For $\sigma_{22}^{\infty}=-\sigma_{11}^{\infty}=\sigma_0$, the analytic functions in the N phases take the following simple forms

$$\varphi_k(z)=c_1^{(k)}z^3+c_2^{(k)}z^{-1},\quad \psi_k(z)=c_3^{(k)}z+c_4^{(k)}z^{-3},\quad k=1,2,\ldots,N,\tag{5}$$

where $c_1^{(k)}, c_2^{(k)}, c_3^{(k)}$ and $c_4^{(k)}$ are unknown time-dependent real coefficients to be determined.

By enforcing the interface conditions in Eq. (4), we arrive at

$$\begin{aligned}
&\chi_j\Gamma_j\left[\kappa_{j+1}\tilde{R}_j^2 \quad \tilde{R}_j^{-2} \quad 0 \quad -\tilde{R}_j^{-4}\right]\dot{\mathbf{y}}_{j+1}+\gamma_j\Gamma_j\left[-3\tilde{R}_j^2 \quad \kappa_{j+1}\tilde{R}_j^{-2} \quad -1 \quad 0\right]\dot{\mathbf{y}}_{j+1}\\
&-\chi_j\left[\kappa_j\tilde{R}_j^2 \quad \tilde{R}_j^{-2} \quad 0 \quad -\tilde{R}_j^{-4}\right]\dot{\mathbf{y}}_j-\gamma_j\left[-3\tilde{R}_j^2 \quad \kappa_j\tilde{R}_j^{-2} \quad -1 \quad 0\right]\dot{\mathbf{y}}_j\\
&=3\left[\tilde{R}_j^2 \quad -\tilde{R}_j^{-2} \quad 0 \quad \tilde{R}_j^{-4}\right]\mathbf{y}_j,\\
&\gamma_j\Gamma_j\left[\kappa_{j+1}\tilde{R}_j^2 \quad \tilde{R}_j^{-2} \quad 0 \quad -\tilde{R}_j^{-4}\right]\dot{\mathbf{y}}_{j+1}+\chi_j\Gamma_j\left[-3\tilde{R}_j^2 \quad \kappa_{j+1}\tilde{R}_j^{-2} \quad -1 \quad 0\right]\dot{\mathbf{y}}_{j+1}\\
&-\gamma_j\left[\kappa_j\tilde{R}_j^2 \quad \tilde{R}_j^{-2} \quad 0 \quad -\tilde{R}_j^{-4}\right]\dot{\mathbf{y}}_j-\chi_j\left[-3\tilde{R}_j^2 \quad \kappa_j\tilde{R}_j^{-2} \quad -1 \quad 0\right]\dot{\mathbf{y}}_j\\
&=-\left[3\tilde{R}_j^2 \quad \tilde{R}_j^{-2} \quad 1 \quad 0\right]\mathbf{y}_j,\\
&\left[\tilde{R}_j^2 \quad -\tilde{R}_j^{-2} \quad 0 \quad \tilde{R}_j^{-4}\right](\mathbf{y}_j-\mathbf{y}_{j+1})=\mathbf{0},\\
&\left[3\tilde{R}_j^2 \quad \tilde{R}_j^{-2} \quad 1 \quad 0\right](\mathbf{y}_j-\mathbf{y}_{j+1})=\mathbf{0},\ j=1,2,\ldots,N-1,
\end{aligned}\tag{6}$$