International Journal of Engineering Science 120 (2017) 155–171

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

A note on stress/strain conjugate pairs: Explicit and implicit theories of thermoelasticity for anisotropic materials

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ARTICLE INFO

Article history:
Received 9 December 2016
Revised 14 July 2017
Accepted 9 August 2017

MSC:
15A06
65F05
74A05
74A10
74A15
74A20

Keywords:
Distortion
Encoding/decoding maps
Dilatation
Extension
Upper-triangular kinematics

ABSTRACT

A set of six, independent, stress/strain, conjugate pairs are derived: One for dilation, two for squeeze, and three for shear. They follow from a Gram–Schmidt factorization of the deformation gradient. Theories for elastic solids are derived in terms of these conjugate pairs. Anisotropy is introduced through bijective maps between tensor components and constituents of the basis. The boundary value problems of simple tension and uniform pressure are used to illustrate the effects of anisotropy, as predicted by a Hooke-like material model.

1. Introduction

In 2012, Srinivasa (2012) considered a Gram–Schmidt decomposition of the deformation gradient $\mathbf{F}$ (an $\mathbf{QR}$ factorization of $\mathbf{F}$ that he denoted as $\mathbf{Q}\tilde{\mathbf{F}}$) in place of the polar decomposition (an $\mathbf{RU}$ factorization of $\mathbf{F}$) commonly adopted throughout the mechanics literature. Among the various features that a Gram–Schmidt factorization of the gradient of deformation produces is a coordinate frame that is 'nearly' embedded within the material of interest. It is nearly embedded in that a truly embedded coordinate frame would be curvilinear with each coordinate curve being comprised of the same set of material particles over time (Oldroyd, 1950), whereas, here the coordinate frame is taken to be rectangular Cartesian. In the coordinate frame associated with a Gram–Schmidt factorization of $\mathbf{F}$, the 1 coordinate direction remains tangent to a 1 material curve at the origin, and the 12 coordinate plane remains tangent to a 12 material surface, cf. Srinivasa (2012, Fig. 1) and Freed and Srinivasa (2015, Fig. 1). As a student of Lodge (1964; 1974), this property intrigued me. This is also an attractive property for experimentalists. This paper exploits some of the theoretical advantages that a triangular distortion tensor (arising from a Gram–Schmidt decomposition of $\mathbf{F}$) has over a symmetric stretch tensor (arising from a polar decomposition $\mathbf{F}$) when constructing material models.

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http://dx.doi.org/10.1016/j.ijengsci.2017.08.002

Tensor invariants (Rivlin & Smith, 1969; Spencer, 1971) are commonly employed in the construction of constitutive theories throughout mechanics. Although the theory is elegant, putting it into practice is not always straightforward. Rivlin and Saunders (1951) were the first to experience the primary weakness of this theoretical approach: Covariance between invariants hampers one's ability to parameterize material models derived from such a theoretical structure. This deficiency was laid bare by Criscione (2003) and an alternative paradigm was put forward (Criscione, Douglas, & Hunter, 2001; Criscione, Humphrey, Douglas, & Hunter, 2000; Criscione, Sacks, & Hunter, 2003a; 2003b): Construct an orthogonal set of strain invariants to use in place of those that come from the Cayley–Hamilton theorem. Adaptation of his approach has been hampered because of implementation challenges from the perspective of an experimentalist. This paper presents yet another attempt to provide a theoretical foundation for the construction of material models, hopefully overcoming some of the deficiencies of these prior approaches.

Here the notion that a set of scalar-valued strain invariants used to describe the state of a material is replaced with the premise that a set of scalar-valued, conjugate, stress/strain, base pairs, i.e., a basis of thermodynamic origin, provides a more apt description for the state of a material. The author's stress/strain base pairs are an example of Hill's (1972,1978) generalized tractions/displacements, which provide a coordinate (frame) invariant description of the work being done on a deformable element of mass. The author demonstrates that such a basis exists. Theories of elasticity are derived in terms of this basis. Bijective maps are constructed that convert tensor components into base pairs and vice versa. These maps take tensor fields describing an anisotropic material response and map them into a stress/strain basis belonging to an equivalent uniform material. Anisotropy is introduced through the map. This idea is analogous to how Karafillis and Boyce (1993) map the yield surface for an anisotropic metal into a plastically equivalent isotropic material.

The paper is organized in the following manner. The kinematics described by a **QR** (Gram–Schmidt) factorization of the deformation gradient $\mathbf{F}$ are reviewed, with fields for a deformation gradient, velocity gradient, and spin being established for this rotated frame of Gram–Schmidt; they are the outputs of Algorithm 2. The coordinate frame arising from a Gram–Schmidt factorization of the deformation gradient is referred to as the experimentor's frame, because it is within this frame of reference that the physical components of distortion can be directly observed and readily measured (Freed, Erel, & Moreno, 2017; Freed & Srinivasa, 2015; Srinivasa, 2012).

The fact that the rate of distortion can be decomposed into three additive parts, independent of constitution, implies that the stress power can be decomposed in like manner, independent of constitution. This decomposition leads to six, independent, stress/strain, base pairs that are thermodynamic conjugates to one another: One for dilation, two for squeeze, and three for shear. Bijective maps are constructed that convert tensor components into these stress/strain-rate base pairs. These maps are not unique. They are modified to account for material anisotropy. At this juncture, explicit and implicit elastic theories are derived from their respective energy functions. Uniaxial extension of a Hooke-like material in three coordinate directions along with the case of uniform pressure illustrate how anisotropy manifests itself in this theoretical approach. A two-dimensional version of the theory has been successfully applied to composite panels (Erel & Freed, 2017). The paper concludes with an overview of the theory.

2. Kinematics

Srinivasa (2012) has shown that the 3D distortion tensor $\tilde{\mathbf{F}} = \mathbf{Q}^{\mathsf{T}}\mathbf{F}$ derived from a **QR** factorization of the deformation gradient $\mathbf{F}$, with $\mathbf{Q}$ being a proper orthogonal tensor, populates an upper-triangular matrix with components acquired from a Cholesky factorization (Freed, 1998) of the Green deformation tensor $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$. They are assigned accordingly to Gram's decomposition procedure, which is applicable for any matrix with a positive determinant, so that for the case of a gradient in deformation one has

$$
\begin{aligned}
&\tilde{F}_{11} = \sqrt{C_{11}} && \tilde{F}_{12} = \frac{C_{12}}{\tilde{F}_{11}} && \tilde{F}_{13} = \frac{C_{13}}{\tilde{F}_{11}} \\
&\tilde{F}_{21} = 0 && \tilde{F}_{22} = \sqrt{C_{22} - \tilde{F}_{12}^2} && \tilde{F}_{23} = \frac{C_{23} - \tilde{F}_{12}\tilde{F}_{13}}{\tilde{F}_{22}} \\
&\tilde{F}_{31} = 0 && \tilde{F}_{32} = 0 && \tilde{F}_{33} = \sqrt{C_{33} - \tilde{F}_{13}^2 - \tilde{F}_{23}^2}
\end{aligned}
\tag{1}
$$

which have been shown to decompose into three fundamental modes, viz.,

$$
[\tilde{\mathbf{F}}] = [\tilde{\boldsymbol{\Lambda}}][\tilde{\mathbf{F}}^{\alpha\beta}][\tilde{\mathbf{F}}^{\gamma}] = \underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}}_{\text{extension}} \times \underbrace{\overbrace{\begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix}}^{\text{out-of-plane}} \times \overbrace{\begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{in-plane}}}_{\text{shear}} = \begin{bmatrix} a & a\gamma & a\beta \\ 0 & b & b\alpha \\ 0 & 0 & c \end{bmatrix}
\tag{2}
$$

whose elements of extension and shear have physical components defined by

$$
a := \tilde{F}_{11}, \quad b := \tilde{F}_{22}, \quad c := \tilde{F}_{33}, \quad \alpha := \frac{\tilde{F}_{23}}{\tilde{F}_{22}}, \quad \beta := \frac{\tilde{F}_{13}}{\tilde{F}_{11}}, \quad \gamma := \frac{\tilde{F}_{12}}{\tilde{F}_{11}}
\tag{3}
$$