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An asymptotically exact theory of functionally graded piezoelectric shells

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ABSTRACT

An asymptotically exact two-dimensional theory of functionally graded piezoelectric shells is derived by the variational-asymptotic method. The error estimation of the constructed theory is given in the energetic norm. As an application, analytical solution to the problem of forced vibration of a functionally graded piezoceramic cylindrical shell with thickness polarization fully covered by electrodes and excited by a harmonic voltage is found.

1. Introduction

Functionally graded materials (FGM) were first invented by a group of Japanese scientists (Koizumi, 1992; Niino & Maeda, 1990) and have been since then widely used in smart structures for the active vibration control (He, Ng, Sivashanker, & Liew, 2001; Jha, Kant, & Singh, 2013). Such smart structures in form of plates or shells are quite often made of the functionally graded piezoelectric (FGP) materials whose electroelastic moduli vary through the thickness (Wu, Kahn, & Moy, 1996). If oscillating voltages, as external excitations, are controlled on electrodes covering faces or boundaries of such structures, then, under certain conditions, the structures may exhibit anti-resonant regime that can be used to eliminate unwanted vibrations (Preumont, 2002). Mention that, from the formal mathematical point of view, smart sandwich structures with piezo patches bonded to elastic layers (Bailey & Ubbard, 1985; Crawley & De Luis, 1987; Crawley & Lazarus, 1991; Tzou & Gadre, 1989) also belong to the FGP-structures with piecewise constant electroelastic moduli.

Due to the above mentioned inhomogeneous material properties of FGP-structures, the problems of their equilibrium and vibration admit exact analytical solutions of the three-dimensional theory of piezoelectricity only in a few exceptional cases (see, e.g., Elishakoff, Pentaras, & Gentilini, 2015; Pan & Han, 2005; Vel & Batra, 2004; Zhong & Shang, 2003 and the references therein). By this reason different approaches have been developed depending on the type of the structures. If FGP-plates and shells are thick, no accurate two-dimensional theory can be constructed, so only the numerical methods or semi-analytical methods applied to three-dimensional theory of piezoelectricity make sense (Dong, 2008; Li, Iu, & Kou, 2008; Wen, Sladek, & Sladek, 2011). However, if FGP-plates and shells are thin, the reduction from the three- to two-dimensional theory is possible and different approximations can be constructed. Up to now two main approaches have been developed: (i) the variational approach based on Hamilton's variational principle and on some ad-hoc assumptions gener-

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alizing Kirchhoff–Love's hypothesis to FGP-plates and shells (Han, Liu, Xi, & Lam, 2001; He et al., 2001; Wu, Chen, Shen, & Tian, 2002; Yiqi & Yiming, 2010),[1] (ii) the asymptotic approach based on the analysis of the three-dimensional equations of piezoelectricity, mainly for the laminated FGP-plates (Cheng & Batra, 2000; Leugering, Nazarov, & Slutskij, 2012; Wu & Tsai, 2009). The disadvantage of the variational approach is the necessity of having an Ansatz for the displacements and electric field that is difficult to be justified, while simplicity and brevity are its advantages. The asymptotic method needs no a priori assumptions; however, the direct asymptotic analysis of the 3-D differential equations of piezoelectricity is very cumbersome. The synthesis of these two approaches, called the variational-asymptotic method, first proposed by Berdichevsky (1979) and developed further by Le (1999) for the dynamic case, seems to avoid the disadvantages of both approaches described above and proved to be quite effective in constructing approximate equations for thin-walled structures. Note that this method has been applied, among others, to derive the 2-D theory of homogeneous piezoelectric shells by Le (1984, 1986b), the 2-D theory of purely elastic anisotropic and inhomogeneous shells (Berdichevsky, 2009), the 2-D theory of purely elastic sandwich plates and shells by Berdichevsky (2010a, 2010b), the theory of smart beams by Roy, Yu, and Han (2007), the theory of low- and high frequency vibration of laminate composite shells by Lee and Hodges (2009a, 2009b), and just recently, the theory of smart sandwich shells by Le and Yi (2016). Note also the closely related method of gamma convergence used in homogenization of periodic and random microstructures (Braides, 2002) and dimension reduction of plate theories (Friesecke, James, & Müller, 2006).

The aim of this paper is to construct the rigorous first order approximate two-dimensional FGP-shell theory by the variational-asymptotic method. We consider the FGP-shell whose electroelastic moduli vary in the thickness direction. The dimension reduction is based on the asymptotic analysis of the action functional containing small parameters that enables one to find the distribution of the displacements and electric field from the solution of the so-called thickness problem. Using the generalized Prager–Synge identity for the inhomogeneous piezoelectric body, we provide also the error estimation of the constructed theory in the energetic norm. We apply this theory to the problem of forced vibration of a functionally graded piezoceramic cylindrical shell with thickness polarization fully covered by electrodes and excited by a harmonic voltage. The exact analytical solution to this problem is found.

The paper is organized as follows. After this short introduction the variational formulation of the problem is given in Sections 2 and 3. Sections 4,5 are devoted to the asymptotic analysis of the action functional. In Section 6 the two-dimensional theory of FGP-shells is obtained. Section 7 provides the error estimation of the constructed theory. Section 8 presents the exact analytical solution to the problem of forced vibration of a circular cylindrical FGP-shell. Finally, Section 9 concludes the paper.

2. Variational principle of piezoelectricity

Let $\mathcal{V} \subseteq \mathbb{R}^3$ be a domain of the three-dimensional euclidean space occupied by a linear and inhomogeneous piezoelectric body in its stress-free undeformed state. A motion of this body is completely determined by two fields, namely, the displacement field $\mathbf{w}(\mathbf{x}, t)$, and the electric potential $\varphi(\mathbf{x}, t)$. For simplicity we will consider the case of purely electrical loading on the body, which corresponds to specifying the value of the electric potential on the electrodes. Let the boundary of the body, $\partial\mathcal{V}$, be decomposed into $n+1$ two-dimensional surfaces $\mathcal{S}_e^{(1)}, \dots, \mathcal{S}_e^{(n)}$, and $\mathcal{S}_d$. The subboundaries $\mathcal{S}_e^{(1)}, \dots, \mathcal{S}_e^{(n)}$ are covered by electrodes. We assume that the electrodes are infinitely thin so that their kinetic and electroelastic energies can be neglected compared with those of the body. On these electrodes the electric potential is prescribed

$$\varphi = \varphi_{(i)}(t) \quad \text{on } \mathcal{S}_e^{(i)}, i = 1, \dots, n. \tag{1}$$

Hamilton's variational principle of piezoelectricity (see, e.g., Le, 1986b; 1999) states that the true displacement $\check{\mathbf{w}}(\mathbf{x}, t)$ and electric potential $\check{\varphi}(\mathbf{x}, t)$ of an inhomogeneous piezoelectric body change in space and time in such a way that the action functional

$$I[\mathbf{w}(\mathbf{x}, t), \varphi(\mathbf{x}, t)] = \int_{t_0}^{t_1} \int_{\mathcal{V}} [T(\mathbf{x}, \dot{\mathbf{w}}) - W(\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{E})]\, dv\, dt \tag{2}$$

becomes stationary among all continuously differentiable functions $\mathbf{w}(\mathbf{x}, t)$ and $\varphi(\mathbf{x}, t)$ satisfying the initial and end conditions

$$\mathbf{w}(\mathbf{x}, t_0) = \mathbf{w}_0(\mathbf{x}), \quad \mathbf{w}(\mathbf{x}, t_1) = \mathbf{w}_1(\mathbf{x}),$$

as well as constraints (1). The integrand in the action functional (2) is called Lagrangian, while dv is the volume element and the dot over quantities denotes the partial time derivative. In the Lagrangian $T(\mathbf{x}, \dot{\mathbf{w}})$ describes the kinetic energy density given by[2]

$$T(\mathbf{x}, \dot{\mathbf{w}}) = \frac{1}{2}\rho(\mathbf{x})\dot{\mathbf{w}} \cdot \dot{\mathbf{w}},$$

[1] The literature on this topic is huge due to the variety of the 2-D FGP-shell and plate theories: single-layer, multi-layer, refined theories including rotary inertias and transverse shears et cetera. It is therefore impossible to cite all references. For the overview the reader may consult Reddy (2004),Liew, Zhao, and Ferreira (2011),Shen (2016) and the references therein.

[2] As we shall be concerned with mechanical vibrations of non-conducting piezoelectric bodies at frequencies far below optical frequencies, the coupling between the electric and magnetic fields and the dependence of the kinetic energy on $\dot{\varphi}$ can be neglected.