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A thermodynamically consistent formulation of generalized thermoelasticity at finite deformations

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ABSTRACT

A thermodynamically consistent model of non-classical coupled non-linear thermoelasticity capable of accounting for thermal wave propagation is proposed. The heat flux is assumed to consist of both additive energetic and dissipative components. Constitutive relations for the stress, the entropy and the energetic component of the heat flux are derived in a thermodynamically consistent manner. A Lyapunov function for the dynamics is obtained for the case in which the surface of the continuum body is maintained at a reference temperature. It is shown that the system is non-linearly stable. The linearized model is shown to be similar to the type III model of Green and Naghdi, except for some minor differences in the interpretations of some of the parameters.

1. Introduction

The propagation of heat energy in a rigid body is governed by the energy balance equation which reads

$$\dot{e} = -\operatorname{div}\mathbf{q} + r. \tag{1}$$

Here e, $\mathbf{q}$, and r denote the internal energy, heat flux, and heat source respectively. The superposed dot denotes time derivative and div[·] divergence. In the classical theory of heat conduction with Fourier's law (Fourier, 1822), the balance Eq. (1) is supplemented by a constitutive equation for the heat flux $\mathbf{q}$ satisfying the inequality

$$\mathbf{q}\cdot\nabla\Theta \le 0, \tag{2}$$

where Θ is the absolute temperature. The inequality (2), also referred to as *heat conduction inequality*, states that heat only flows by the mechanism of diffusion from regions of higher to those of lower temperature distributions. In other words, the vector quantity describing the heat flow, that is, the heat flux $\mathbf{q}$, always points in the direction of decreasing temperature distribution.

However, for example, in the linear case this mechanism leads to a parabolic partial differential equation which is characterized by infinite speed of propagation of localized thermal disturbances, a paradoxical phenomenon from a physical point of view.

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Despite this non-physical prediction, the classical theory of heat conduction by Fourier's law has been successful for a broad range of engineering applications. However, as length scales decrease, this phenomenon of instantaneous propagation of thermal disturbances becomes more dominant at low ranges of temperature near absolute zero, so that approximations used in the classical theory of heat conduction lose validity (Fryer & Struchtrup, 2013).

The earliest known conjecture on the existence of thermal propagation as waves, also known as the second sound phenomenon, was given by Nernst (1918) in 1917. Later, in Tisza (1938) and Landau (1941) in 1941 independently suggested the possibility of thermal waves in superfluid liquid helium, at temperatures below the so-called lambda transition near 2.2 K (Donnelly, 2009). Peshkov (1944) reported the first experimental evidence for the existence of second sound in 2He. In his work, Peshkov suggests that second sound might also be observed in pure crystalline materials on the basis of similarities between crystal materials and liquid helium. Laser pulsing experiments have shown that second sound can propagate in high-purity crystals of 4He (Ackerman, Bertman, Fairbank, & Guyer, 1966), 3He (Ackerman & Overton, 1969), NaF (Jackson, Walker, & McNelly, 1970), and Bi (Narayanamurti & Dynes, 1972).

Cattaneo (1958) was the first to introduce a non-Fourier theory of heat conduction in order to overcome the paradoxical prediction of the classical Fourier theory. His work is based on the concept of relaxing the heat flux from the classical Fourier law to obtain a constitutive relation with a non-Fourier effect. There have been several other attempts to develop continuum theories capable of predicting thermal waves propagating at finite speeds for various types of media. Among these, the works in Lord and Shulman (1967), Gurtin and Pipkin (1968), Fox (1969), Müller (1971) and Green and Lindsay (1972) are notable. The constitutive equation for the heat flux according to Cattaneo is examined in terms of thermodynamics in Coleman, Fabrizio, and Owen (1982a,b). Later this result was extended in Öncü and Moodie (1992) to the case where deformation is applicable.

A relatively more recent theory of non-classical heat conduction with and without deformation was proposed by Green and Naghdi (1991,1992,1993,1995). Their work is based on the introduction of three types of constitutive relation for the heat flux, thereby resulting in three different models, namely type I, which is the classical theory, type II, a purely hyperbolic model which allows the propagation of a heat pulse without damping, and type III, which is the combination of the first two.

The physical basis for and evidence of second sound is the subject of considerable debate (Bright & Zhang, 2009). Nonetheless, non-classical models of heat conduction are capable of reproducing features that have also been observed in some of the literature. It is also prototypical of other problems such as case II diffusion (see the articles Bargmann, McBride, and Steinmann, 2011; Govindjee and Simo, 1993 and the references, for details).

In recent years there has been a considerable amount of interest concerning the theory of Green and Naghdi. An extensive overview of the theory can be found in Chandrasekharaiah (1996,1998) and Richard and Hetnarski (1999). Theoretical results addressing existence and uniqueness (Quintanilla, 2002a, 2002b) and exponential stability (Racke, 2002) have also been investigated for some types of the theory. The designing of appropriate numerical methods has also been addressed in Bargmann and Steinmann (2006), Wakeni, Reddy, and McBride.

The non-classical theory of Green and Naghdi relies on the introduction of an internal state variable, the so called *thermal displacement*, which is a time primitive of the temperature field. The role and interpretation of the thermal displacement have generated substantial discussion in the literature. The first use of the concept of thermal displacement dates back to 1884 when Helmoltz utilized it to construct the Lagrangian of a mechanical system regarding it as a coordinate which appear only through its time derivative, see for example, Podio-Guidugli (2009) and the references therein. Other works on its use include, for example, models for dissipative materials and relativistic perfect fluids are also reviewed in the aforementioned paper. We refer also to the work by Dascalu and Kalpakides (2005), who inter alia emphasize the importance of the thermal displacement in arriving at a consistent theory of thermoelastic crack propagation.

This work is concerned with a thermodynamically consistent formulation of a fully non-linear coupled problem of non-classical thermoelasticity inspired by that of Green and Naghdi. The formulation is based on the basic laws of continuum thermodynamics, the balance laws of momentum, the balance of energy, and the entropy imbalance. However, the point of departure from the classical theory comes from two assumptions: the first is that the heat flux is additively composed of two parts, namely the dissipative and energetic components, and the second is that a material derivative of a time primitive of the absolute temperature is assumed to be the proportionality constant of the heat and the entropy conjugate pairs.

Thermodynamic restrictions on the constitutive relations are derived using the procedure of Coleman and Noll (1963). Stability of the system of partial differential equations governing the thermomechanical coupling in the non-classical regime is proved in the sense of Lyapunov. The other notable aspect of the model is that the linearized theory is similar to that of by Green and Naghdi except for some differences in the interpretation of the material parameters.

The rest of the paper is organized as follows. In Section 2, geometric and kinematical descriptions of the continuum body are presented. In Section 3 the non-classical model describing the coupling of mechanical deformation and non-classical heat conduction is formulated based on the laws of thermodynamics. Constitutive relations for the stress, the entropy and the energetic component of the heat flux are derived from a free energy via the Coleman–Noll procedure. Next, in Section 4 the initial boundary value problem (IBVP) of non-classical thermoelasticity is summarized. A class of physically meaningful initial and boundary conditions are also proposed. A Lyapunov function for the dynamics generated by the IBVP rendering the system non-linearly stable is obtained in Section 5. Finally in Section 6, the linearized form of the IBVP is summarized.