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# On the concept of “far points” in Hertz contact problems

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ABSTRACT

Relative displacement of “far points” is used in the Hertzian contact mechanics as a measure of contact compliance. However, to be legitimate, it should be almost insensitive to the exact choice of the “far points”, and this is not always the case. The present work aims at examination of legitimacy of this concept, on specific examples of one-dimensional problem of a long rod, 2-D problem of heavy disk and 3-D problem of a sphere resting on a smooth rigid foundation. It is found that, whereas in the 1-D problem this concept may well become inadequate, in the considered 2-D and 3-D problems, the parameter controlling the legitimacy of this concept are identified and, in the vast majority of cases of practical interest, the concept is indeed legitimate. Note that the mentioned 2-D and 3-D problems are quite challenging and the presented solutions may be of interest of their own.



## 1. Introduction

A small contact area between two elastic bodies creates a highly compliant zone in the vicinity of the contact: when the bodies are pressed against one another, most of the overall deformation comes from this zone. The contact zone compliance is usually characterized by approach of two “remote points” – points on two sides of the contact that are sufficiently far from the zone (Hertz, 1881; Johnson, 1985). Relative displacement (approach) of “far points” on two sides of a contact is one of the quantities of interest in Hertzian contact problems. It is used, for example, as a measure of the contact compliance, the underlying idea being that the dominant contribution to this displacement comes from the contact zone, and the contribution of remaining parts of the contacting bodies is negligible.

For the approach of “far points” to be a legitimate measure of contact compliance, it should be almost insensitive to the exact choice of these points. This is not immediately obvious, and an example can be easily given where the concept fails. If one, or both, of the contacting bodies have elongated shapes in the direction normal to the contact plane, displacements accumulated in them may be comparable to the contribution of the contact zone. The following simple problem illustrates this statement. Consider a long elastic rod with rounded end, of length $2L$ and cross-section radius $r$, pressed against a rigid wall by force $P$ applied at the opposite end; the contact with the wall is circular, of radius $a$ (Fig. 1).

According to the Hertzian theory, a contribution to the approach of the rod’s center (point $A$) towards the wall generated by the convex contact zone equals to $\delta = \frac{1-\nu}{4a\mu}P$, where $\mu$ and $\nu$ are the shear modulus and Poisson’s ratio, respectively. We now compare $\delta$ with the displacement $u_L$ accumulated in the rod at the distance $L$ from the contact due to the longitudinal

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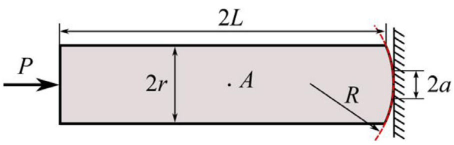


**Fig. 1.** A long elastic rod forming contact of radius $a$ with a rigid wall.

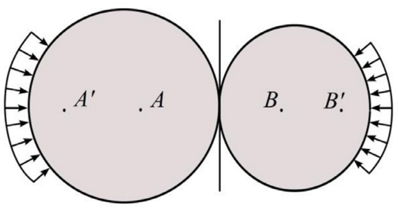


**Fig. 2.** Two elastic bodies in contact.

deformation of the rod. For an approximate estimate, we assume, using the Saint-Venant's principle, that, somewhat away from the contact, the rod experiences uniform compression; then $u_L = \frac{L}{\pi r^2 E} P$, where $E$ is Young's modulus. For the inequality $u_L < <\delta$ to hold, the length $L$ should not be too large, namely, we must have

$$\frac{a}{r} \ll c(\nu)\frac{r}{L} \tag{1.1}$$

where $c(\nu) = \pi(1-\nu^2)/2$ is a constant, which changes between 1.75 and 1.18 for $\nu$ varying from 0 to 0.5.

For example, if the aspect ratio of the rod $L/r$ is 10, then $a < <r/10$, i.e., practically speaking, $r$ should be of the order of $10a$ or grater. Otherwise, the displacement from the deformation of the rod accumulated at distance $L$ is non-negligible compared to the displacement generated by the contact zone, so that the displacement $u_L$ at the point $A$ cannot be used as a measure of contact compliance in the Hertzian contact theory. Observe also that $a = \sqrt{\delta R}$, and therefore, with increasing radius $R$ of the end-rounding, the contribution from the longitudinal deformation increases as well. Note that if the "pencil-like" body is positioned vertically and is pressed against a rigid floor by gravitational forces, we have the same inequality (1.1), with somewhat different constant, of $c(\nu) = \pi(1-\nu^2)/3$.

In the above example, the failure of the concept of "far points' is related to the elongated geometry of the elastic body. However, the mentioned insensitivity to the choice of "remote points" may, possibly, be violated even for solids of non-elongated shapes. In the text to follow, we examine this issue on two example problems: a heavy 2-D disk and 3-D sphere resting on a rigid frictionless foundation that deforming under their own weight. We show that, although in most cases of practical interest the insensitivity does hold, for certain combinations of the elastic modulus and the specific weight it may be violated; these combinations will be identified in the solution obtained in the text to follow.

In a general setting, we consider two contacting bodies, and choose two points, $A$ and $A$' belonging to the first one (Fig. 2) that are sufficiently far from the contact plane and thus can be regarded as "far points". Here "sufficiently far" means that the distance from the point to the contact plane is much greater than the characteristic size of the contact area. Point $A$' is substantially farther away from the contact plane than $A$ (the distance between them, in the direction normal to the plane, is comparable to the size of the contacting body). The concept of "far points" can be considered legitimate if displacements of these points in the direction towards the contact plane obey the inequality

$$\frac{\left|u(A') - u(A)\right|}{|u(A)|} << 1 \tag{1.2}$$

Similar inequality must hold for points $B$ and $B$' of the second contacting body.

We examine the criterion (1.2) on two examples, a 2-D heavy elastic disk and a 3-D heavy elastic sphere that rest on a rigid frictionless foundation. Note that these problems do not seem to have been solved in literature, and may be of interest of their own. We construct, by employing the method of matched asymptotic expansions, the approximations to the displacement fields away from the contact zone. We identify the parameter that controls legitimacy of the far-points concept, which implicitly assumes that the choice of the "remote points" is unimportant, as long as they are sufficiently far from the contact, i.e. that their approach is relatively insensitive to the choice.

In particular, it will be shown that, if the points $A$ and $A$' are chosen as the center of the considered body and the point at the top of it, then the following inequalities hold:

$$\frac{\left|u(A') - u(A)\right|}{|u(A)|} \leq \frac{0.76}{1.26 + \ln M} \quad (2-\mathrm{D\,case}) \tag{1.3}$$