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On the interior regularity criteria for liquid crystal flows

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ABSTRACT

In this paper, we provide some new regularity criteria for suitable weak solutions to the 3D nematic liquid crystal flow in terms of the velocity field. More precisely, we prove that the suitable weak solution (u, d) is regular provided that either the scaled $L^{p,q}$ norm of u, or the scaled $L^{p,q}$ norm of $\nabla \times u$ is sufficiently small.

1. Introduction

Liquid crystals are substances that exhibit a phase of matter that has properties between those of a conventional liquid, and those of a solid crystal. For instance, a liquid crystal may flow like a liquid, but its molecules may be oriented in a crystal-like way. There are three main types of liquid crystals: nematic, termed smectic and cholesteric. One of the most common liquid crystal phase is the nematic, where the molecules have no positional order, but they have long-range orientational order. The three-dimensional flows of nematic liquid crystals can be governed by the following system of partial differential equations [1–4]:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla \pi - \nabla \cdot (\nabla d \odot \nabla d), \\ \partial_t d + (u \cdot \nabla) d = \gamma(\Delta d - f(d)), \\ (u, d)|_{t=0} = (u_0, d_0), \\ \nabla \cdot u = 0. \end{cases} \tag{1.1}$$

Here $u : \mathbb{R}^3 \times (0, +\infty) \to \mathbb{R}^3$ represents the velocity field of the flow, $\pi : \mathbb{R}^3 \times (0, +\infty) \to \mathbb{R}$ represents the pressure of the flow and $d : \mathbb{R}^3 \times (0, +\infty) \to \mathbb{R}^3$ represents the optical molecule direction. (u_0, d_0) is a given initial date, and ν, λ and γ are positive constants. The notation $\nabla d \odot \nabla d$ denotes the 3×3 matrix whose

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(i,j)th entry is given by

$$\partial_i d \cdot \partial_j d \quad (1 \le i,j \le 3),$$

and $f(d) = \frac{1}{\eta^2}(|d|^2 - 1)d$, the Ginzburg–Landau approximation function, is smooth and bounded function, where η is a positive constant.

The system (1.1) was first introduced by Lin [2] as a simplification to the Ericksen–Leslie system describing the flow of nematic liquid crystals. Although system (1.1) is a simplified version, it still retains most of the essential features of the Ericksen–Leslie equation for the hydrodynamics of nematic liquid crystal. In [4,5], Lin–Liu gave a systematic mathematical analysis and obtained the following existence and partial regularity results:

- For $u_0 \in L^2(\mathbb{R}^3)$ and $d_0 \in H^1(\mathbb{R}^3)$, there exists a global weak solution (u,d) of (1.1) satisfying

$$\begin{cases} u \in L^2(0,T;H^1(\mathbb{R}^3)) \cap L^\infty(0,T;L^2(\mathbb{R}^3)), \\ d \in L^2(0,T;H^2(\mathbb{R}^3)) \cap L^\infty(0,T;H^1(\mathbb{R}^3)), \quad \forall T \in (0,+\infty). \end{cases}$$

- There exists a unique global classical solution (u,d) to (1.1) provided $(u_0,d_0) \in H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ and either $N=2$ or $N=3$ and $\nu \ge \nu(u_0,d_0)$.
- The one-dimensional spacetime Hausdorff measure of the singular set of the so-called suitable weak solution (u,d) is zero. Here by suitable weak solutions we mean solutions that solve (1.1) in the sense of distribution and satisfy the local energy inequality (see Definition 1.1 for details).

In particular, they proved the following local ε-regularity criteria:

Proposition 1.1 ([5]). *Let (u,d,π) be a suitable weak solution of (1.1) in $Q_1(z_0)$. There exists an $\varepsilon > 0$ such that if*

$$\frac{1}{r^2}\int_{Q_r(z_0)} \left(|u|^3 + |\nabla d|^3 + |\pi|^{\frac{3}{2}}\right) \mathrm{d}x\,\mathrm{d}t < \varepsilon$$

for some $0 < r < 1$, then (u,d) is regular at z_0, i.e., $(u,\nabla d)$ is bounded in $Q_{\frac{r}{2}}(z_0)$. Here $Q_r(z_0)$ is denoted by $B_r(x_0) \times (t_0 - r^2, t_0)$ with $z_0 = (x_0,t_0)$ and

$$B_r(x_0) := \left\{x \in \mathbb{R}^3 : |x - x_0| < r\right\}.$$

Recently, Hu and Wang in [6] established global existence of strong solutions and weak–strong uniqueness under suitable initial conditions. Subsequently, this result was extended by Liu–Zhao–Cui, for details see [7].

In this paper, we consider regularity of suitable weak solutions of (1.1). To illuminate the motivations of this paper in detail, we shall recall some regularity criteria results of Navier–Stokes equations (the orientation field d is not taken into account, i.e., $d = 0$). It is well-known that if the weak solution of the Navier–Stokes equations satisfies the so called Ladyzhenskaya–Prodi–Serrin type condition

$$u \in L^q(0,T;L^p(\mathbb{R}^3)) \;\text{ with }\; \frac{2}{q} + \frac{3}{p} \le 1, \quad p > 3,$$

then it is regular in $\mathbb{R}^3 \times (0,T)$, see [8]. The regularity criteria in the limiting case (i.e., $u \in L^\infty(0,T;L^3(\mathbb{R}^3))$) were solved by Escauriaza, Seregin, and Šverák [9]. In 2007, some different type regularity criteria were obtained by Gustafan–Kang–Tsai [10], one of these regularity criteria says that if there exist $r_0 > 0$ and a small $\varepsilon > 0$ such that

$$\sup_{0<r<r_0} r^{1-\frac{3}{p}-\frac{2}{q}} \|u\|_{L^{p,q}(Q_r(z_0))} < \varepsilon$$