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Detecting an inclusion in an elastic body with residual stress

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ABSTRACT

We consider the inverse problem for estimating the size of an inclusion D, $D \subset \Omega$, in an elastic body with residual stress. The constitutive equation of this elasticity system is not isotropic, due to the presence of residual stresses. We prove that the size of the inclusion can be estimated both from above and below by using only one pair of traction–displacement measurement on the boundary of Ω.

1. Introduction

In this research, the focus is on an inverse problem for the elasticity with residual stresses. The main purpose is to estimate the size of an unknown embedded domain in an elastic body. This embedded domain could represent the region in which the defect occurs. In order to better define the problem, we consider an elastic body with residual stresses. The residual stresses are the remainder after the original cause of the stresses, e.g. thermal treatment, has been removed. The existence of residual stresses may cause premature failure of a structure.

To define our problem more precisely, let Ω be a connected open set in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Assuming that $u(x) = (u_i(x))_{i=1}^3$ is a three-dimensional vector field. We consider the following equilibrium equation for u:

$$\nabla \cdot \sigma = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where $\sigma = (\sigma_{ij})_{i,j=1}^3$ is the stress tensor field given by

$$\sigma(x) = T(x) + (\nabla u)T(x) + \lambda(x)(\mathrm{tr}\widehat{\nabla} u)I + 2\mu(x)\widehat{\nabla} u, \tag{1.2}$$

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where $\widehat{\nabla} u(x) = (\nabla u + \nabla u^t)/2$ is the infinitesimal strain and λ, μ are Lamé parameters. The tensor $T(x) = (t_{jl}(x))_{j,l=1}^3$ represents the residual stress, which satisfies $\nabla \cdot T = 0$ and $t_{jl} = t_{lj}$ for all $1 \leq j, l \leq 3$.

The expression (1.2) is a simple constitutive equation modelling the linear elasticity with residual stress, which has been considered in existing literature [1–3] and [4]. The general equation for linear elasticity with residual stress is given by

$$\sigma = T + (\nabla u)T + L(\widehat{\nabla} u),$$

where $L(\widehat{\nabla} u)$ is the incremental elasticity tensor. The explicit form of $L(\widehat{\nabla} u)$ can be found in [5] and [6]. We consider (1.2) because for (1.1) with (1.2) we have the three spheres inequalities, which are an essential tool in this research.

We can express (1.1) with another format. If we define the elasticity tensor $\mathbf{C} = (\mathbf{C}_{ijkl})_{i,j,k,l=1}^3$ by

$$\mathbf{C}_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + t_{jl}\delta_{ik}, \tag{1.3}$$

then (1.1) is equivalent to

$$\nabla \cdot (\mathbf{C}\nabla u) = \partial_{x_j}(\mathbf{C}_{ijkl}\partial_{x_l} u_k) = 0 \text{ in } \Omega. \tag{1.4}$$

It is rather important to notice that, for this elasticity system, the minor symmetry properties, i.e., $\mathbf{C}_{ijkl} = \mathbf{C}_{jikl}$ and $\mathbf{C}_{ijkl} = \mathbf{C}_{ijlk}$, may not hold. However, it still satisfies the major symmetry property, $\mathbf{C}_{ijkl} = \mathbf{C}_{klij}$, meaning that (1.1) is a hyperelasticity system.

Now let $D \subset \Omega$ represent an unknown domain embedded in Ω. Let $\tilde{\mathbf{C}}$ denote the elasticity tensor in D. We consider the equilibrium system

$$\nabla \cdot ((\chi_{\Omega \setminus D}\mathbf{C} + \chi_D \tilde{\mathbf{C}})\nabla u) = 0 \text{ in } \Omega, \tag{1.5}$$

where χ_E denotes the characteristic function of domain E. Let u be the solution for (1.5) satisfying the Neumann condition

$$(\mathbf{C}\nabla u)\nu = \varphi \text{ on } \partial\Omega, \tag{1.6}$$

where ν is the unit exterior normal to $\partial\Omega$. Here we investigate the following inverse problem: *assuming that the background media* $\mathbf{C}$ *is known, we would like to estimate the size of* D *using the knowledge of* $\{\varphi, u|_{\partial\Omega}\}$ only.

The ultimate goal for this inverse problem is to retrieve all geometric information of D by one pair of $\{\varphi, u|_{\partial\Omega}\}$ only. Detecting size of an inclusion has been studied using various models but yields similar results. We give three significant examples: modelling electrically conducting body [7], modelling the Lamé system of elasticity [8] and modelling the elastic plates [9].

In existing literature, the proof of important result is often based on three spheres inequalities for (1.1), (1.2). The qualitative unique continuation property (UCP) for (1.1), (1.2) has been proved in [1]. Our task here is to derive a quantitative estimate of the UCP and three-sphere inequality for (1.1) and (1.2). The main tool for deriving such quantitative estimate is the Carleman estimate. Unfortunately, we cannot apply the Carleman estimate in [1] directly to our problem here. To overcome this difficulty, we borrow some ideas in [10] to derive the estimates we need. The estimate of $|D|$ is described in Theorem 2.1, which shows that $|D|$ can be bounded both from above and below by the difference of power for the unperturbed system (without D) and the perturbed system (with D) under the fatness condition (Assumption 4 of Section 2). Of course, it is more informative to study the problem without the fatness condition. To do this, we need the quantitative form of the strong unique continuation property (SUCP) for (1.1) and (1.2), i.e. doubling inequalities. However, whether the SUCP holds for (1.1) and (1.2) or not is still an unsolved problem.