Nonlinear Analysis: Real World Applications 40 (2018) 243–259

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

# An exact solution to a Stefan problem with variable thermal conductivity and a Robin boundary condition

Andrea N. Ceretani[a,b,*], Natalia N. Salva[c,d], Domingo A. Tarzia[a]

[a] *CONICET - Depto. de Matemática, Facultad de Ciencias Empresariales, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina*
[b] *Depto. de Matemática, Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Univ. Nacional de Rosario, Pellegrini 250, S2000BTP Rosario, Argentina*
[c] *CONICET - CNEA, Depto. de Mecánica Computacional, Centro Atómico Bariloche, Av. Bustillo 9500, 8400 Bariloche, Argentina*
[d] *Depto. de Matemática, Centro Regional Bariloche, Univ. Nacional del Comahue, Quintral 250, 8400 Bariloche, Argentina*

ARTICLE INFO

*Article history:*
Received 31 August 2017
Accepted 4 September 2017

*Keywords:*
Stefan problems
Exact solutions
Temperature dependent thermal conductivity
Convective boundary conditions
Modified error function
Phase-change processes

ABSTRACT

In this article it is proved the existence of similarity solutions for a one-phase Stefan problem with temperature-dependent thermal conductivity and a Robin condition at the fixed face. The temperature distribution is obtained through a generalized modified error function which is defined as the solution to a nonlinear ordinary differential problem of second order. It is proved that the latter has a unique non-negative bounded analytic solution when the parameter on which it depends assumes small positive values. Moreover, it is shown that the generalized modified error function is concave and increasing, and explicit approximations are proposed for it. Relation between the Stefan problem considered in this article with those with either constant thermal conductivity or a temperature boundary condition is also analysed.



## 1. Introduction

The understanding of phase-change processes has been inspiring scientists from the earlier 18th century. Already in 1831, Lamé and Clapeyron studied problems related to the solidification of the Earth planet [1]. Also the mathematical formulation of phase-change processes as *free boundary problems* dates from the 18th century, since it owes much to the ideas developed by Stefan in 1889 [2–4]. At present, their study is still an active area of research. Besides phase-change processes are interesting in themselves, they attract interests because they are present in a wide variety of situations, both natural and industrial ones. Glass manufacturing and continuous casting of metals are examples of industrial activities involving them, some

* Corresponding author at: CONICET - Depto. de Matemática, Facultad de Ciencias Empresariales, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina.
*E-mail address:* aceretani@austral.edu.ar (A.N. Ceretani).

http://dx.doi.org/10.1016/j.nonrwa.2017.09.002
1468-1218/

recent works in this area are [5,6]. Controlling side-effects of certain industrial processes or preventing future problems derived from our energy-dependent lifestyle, are also examples of how phase-change processes arise as a subject of study [7,8]. Permafrost phenomena or dynamics of snow avalanches are examples of natural situations whose study involves phase-change processes, some recent articles in these subjects are [9–11]. We refer the reader to [12,13] and the references therein for a recent survey in applications and future challenges in free boundary problems. Other references can be seen in the last published Free Boundary Problems International Conference Proceedings [14].

In this article we will focus on phase-change processes that are ensued from an external temperature imposed at some part of the fixed boundary of a homogeneous material. A classical simplification in modelling this sort of phenomena is to consider boundary conditions of Dirichlet type (temperature conditions). This is based on the assumption that heat is instantaneously transferred from the external advise through which a specific temperature is imposed to the material. In view that is physically unrealistic, several authors have suggested to consider conditions of Robin type (convective conditions) since they mimic the fact that the heat transfer at the boundary is proportional to the difference between the imposed temperature and the one the material presents at its boundary (see for example the books [15,16]). Another classical simplification when modelling phase-change processes is to consider that thermophysical properties are constant. Though it is reasonable for most phenomena under moderate temperature variations [15], it is not what actually happens as a rule. In fact, this hypothesis has been removed in many works in the attempt to improve the mathematical model (see, for example [17–20]). All this have encouraged us to look at phase-change processes with convective boundary conditions and non-constant physical properties.

In 1974, Cho and Sunderland studied a phase-change process for a one-dimensional semi-infinite material with temperature-dependent thermal conductivity [21]. The dependence was assumed to be linear, which is a quite good approximation of what actually happens with several materials (water, for example [15]). The phase-change process was assumed to be ensued from a constant temperature imposed at the fixed boundary of the body, what was modelled through a Dirichlet condition. For the resulting Stefan problem, Cho and Sunderland have presented an exact similarity solution. The temperature was obtained through an auxiliary function $\Phi$ that they have called a *Modified Error* (ME) function and that was defined as the solution to a nonlinear ordinary differential problem of second order. Revisiting the work of Cho and Sunderland, a couple of curiosities have arised. On one hand, the existence of the ME function was not proved there. Despite of this lack of theoretical results, the ME function was widely used in the context of phase-change processes before their existence and uniqueness were proved in the recent article [22] (see, for example, [19,23–31]). On the other hand, by following the arguments presented in [21] it is obtained that the ME function must satisfy a differential problem over a closed bounded interval $[0, \lambda]$ with $\Phi(0) = 0$, $\Phi(\lambda) = 1$. Nevertheless, in [21] it was considered a boundary value problem over $[0, +\infty)$ with $\Phi(0) = 0$, $\Phi(+\infty) = 1$. Although in this way it is clearer the relation between the modified and classical error functions (see [21,22] for further details), the change made by Cho and Sunderland add some extra conditions on the temperature function.

In this article we consider a similar phase-change process to that studied in [21]. We are mainly motivated by: (a) improving the modelling of the imposed temperature at the fixed boundary by considering a convective boundary condition, (b) obtaining a solution of similarity type without any extra condition on the temperature distribution. We will study a solidification process, but a completely similar analysis can be done for the case of melting. Aiming for simplicity, we will restrict our presentation to a one-phase process. That is, the case in which the material is initially liquid at its freezing temperature.

The organization of the paper is as follows. First (Section 2), we introduce the one-phase Stefan problem through which we will study the phase-change process. In this section we also present a characterization for any similarity solution to the Stefan problem in terms of a *Generalized Modified Error* (GME) function. This will be defined as the solution to a nonlinear boundary value problem of second order. Similarly to [21], this problem will depend on a positive parameter $\beta$ related to the slope of the thermal conductivity as a linear