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# Evolutionary quasi-variational inequality for a production economy

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A B S T R A C T

This paper is focused on the analysis of a time-dependent general equilibrium problem for a production economy by using the variational inequality approach. More precisely, the time-dependent allocation-price equilibrium consists in searching the optimal allocation by households and firms and in searching the market clearing condition. Firstly, a new formulation of the economic equilibrium problem by means of an evolutionary quasi-variational inequality is proven in the general context of semistrictly quasiconcave utility functions. Finally, an existence result for equilibrium solutions is given.



## 1. Introduction

In this paper, we consider a time-dependent general equilibrium problem for a production economy, where the data evolve in a finite interval of time $[0, T]$. The standard general equilibrium model analyzes market equilibria, i.e., economic situations described by a prices vector and an allocations vector (given by consumption and production). More precisely, the time-dependent allocation-price equilibrium consists in searching for the optimal allocation by households and firms and in searching for the market clearing condition. Households are supposed to maximize their personal utility under economic constraints and firms to maximize their profits under technological constraints. Then, the agents' interaction is modeled in terms of exchanges of goods in the market, imposing that agents' generated market demand and supply are equal. Past literature studies extensively the general equilibrium problem, in the static framework, and we refer the reader to [1] for a survey. In the present paper, the general equilibrium problem is studied in a time-dependent context, by using the variational inequality theory, and assuming generalized concavity and continuity assumptions on utility functions.

The variational inequality theory was developed in the early 1960s by Fichera and Stampacchia in connection with partial differential equations and its power was soon recognized in relation to the study of

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several realistic equilibrium problems which fit in the optimization theory. We refer the reader to recent important contributions [2–11] (with the references therein), where several time-dependent equilibrium problems are studied by means of evolutionary variational inequality arguments.

It is a well-known fact that, if the objective function of an optimization problem is concave and continuously differentiable, then the Stampacchia variational inequality, involving the gradient operator of the objective function, provides a necessary and sufficient optimality condition for a point to be a solution to the optimization problem.

In the optimization theory it is very important to know when a local maximum/minimum is also global and such a useful property is not exclusive to concavity/convexity. See [12] and [13]: in the first paper a class of functions verifying this property is introduced; whereas in the second one, functions which do not satisfy this property are considered. In this context, the class of quasiconcave functions meets a large domain of applications in microeconomics, as well as in nonlinear differential systems. In this setting, the operator involved in the variational problem is replaced by a suitable set-valued map which uses the concept of normal cone (see for more details [14,15]).

This paper aims at introducing a time-dependent model which represents a natural extension of models analyzed, in the static framework, in [16,17] (see also [5]). In this evolutionary market, the aim of each agent (households and firms) is to find the allocation equilibrium, not at the fixed instant $t$, but globally for the whole period of time $[0, T]$. Such equilibrium is obtained mathematically by considering the profit maximization and the maximization of an integral utility function over a budget constraints set. The plan of this paper is the following: Firstly, we describe the time-dependent model of the allocation-price equilibrium for a production economy, in which the integral utility functions are semistrictly quasiconcave and continuous, a natural and classical assumption in mathematical economics. Secondly, we provide a new formulation of the equilibrium in terms of a suitable evolutionary quasi-variational inequality, where the convex constraints set depends on the solution. Finally, we achieve an existence result of the equilibrium points.

## 2. Time-dependent production economy

We consider a general equilibrium problem for an economy involving the production, the exchange and the allocation of resources; these activities take place in a continuous period of time $\mathcal{T} := [0, T]$. In this time-dependent framework, the suitable functional space is the Lebesgue space $L^2(\mathcal{T}, \mathbb{R}^q)$ where for all $F, G \in L^2(\mathcal{T}, \mathbb{R}^q)$, we denote the inner product between $F$ and $G$ by

$$\langle\langle F, G\rangle\rangle_q := \int_{\mathcal{T}} \langle F(t), G(t)\rangle_q \, dt.$$

The marketplace is composed of a finite number $C$ of completely homogeneous commodities, indexed by $c$ and of two types of agents: $H$ households (or consumers), indexed by $h$, and $F$ firms (usually firms and producers have the same meaning), indexed by $f$. We denote by $\mathcal{C} = \{1, \dots, C\}$, $\mathcal{H} = \{1, \dots, H\}$, and $\mathcal{F} = \{1, \dots, F\}$ respectively, the commodities, households and firms sets. Denote by $e_h^c(t)$ and $x_h^c(t)$ the nonnegative quantity of commodity $c$ held and consumed by household $h$ at time $t$ and by $y_f^c(t)$ the quantity of commodity $c$ produced by firm $f$ at time $t$. Denote by $p^c(t)$ the nonnegative price associated to commodity $c$ at time $t$. By grouping the introduced quantities in vectors one has that $e_h(t)$ and $x_h(t)$ are, respectively, the total endowment and the consumption vectors of household $h$, $y_f(t)$ is the production vector of firm $f$ and $p(t)$ is the price vector. We assume that vectors $e_h$ and $x_h$ belong to $L^2(\mathcal{T}, \mathbb{R}^C)$ and the total consumption $x = (x_h)_{h\in\mathcal{H}}$ and the total production $y = (y_f)_{f\in\mathcal{F}}$ of the market belong to $L^2(\mathcal{T}, \mathbb{R}^{CH})$ and $L^2(\mathcal{T}, \mathbb{R}^{CF})$, respectively. We denote by $L$ the space $L^2(\mathcal{T}, \mathbb{R}^C)$ and by $C_L$ the ordering cone of $L$:

$$C_L := \{\alpha \in L : \alpha^c(t) \geq 0 \; a.e. \; in \, \mathcal{T}, \; \forall c \in \mathcal{C}\}.$$