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Viscosity solutions to a model for solid–solid phase transitions driven by material forces

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ABSTRACT

A new phase-field model is formulated to describe martensitic phase transitions driven by material forces, in solid materials, e.g., shape memory alloys. This model is a nonlinear degenerate parabolic equation of second order and its principal part is not in divergence form in multi-dimensional case. The global existence of viscosity solutions to an initial–boundary value problem for this model is proved.

1. Introduction

Martensitic transformations play a key role in materials science, and occur in various materials covering important alloys such as steels, shape memory alloys, see, e.g. [1]. They are displacive, diffusionless and are responsible for the formation of some microstructure, thus determines properties of a material, for example, shape memory effect [1–4]. Martensite, as a result of martensitic transformations, is a crucial microstructure which can grow at temperatures close to absolute zero and at speeds in excess of 1000 ms^{-1}, faster than twice of the speed of sound. Thus by observing this process directly, it is very difficult to obtain helpful information to understand its mechanism, instead mathematical modeling is a powerful tool [5], etc., for instance, phase-field method, though it is quite young, has been proved extremely powerful to both theoretical and numerical analysis of phenomena in materials science, we refer e.g., to [6–10].

As an attempt to understand this type of rapidly changing processes, which thus exists only several micro seconds, H.-D. Alber and the second author of this article have proposed in [11,12] a phase-field model, which is a system of a linear elasticity sub-system coupled to a nonlinear degenerate parabolic equation of

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second order. To formulate this model, we introduce some notations. Let $\Omega \subset \mathbb{R}^3$ be an open domain with smooth boundary $\partial\Omega$. It represents the material points of a solid body. Define $Q_t := (0,t)\times\Omega$. The different phases are characterized by the order parameter $S(t,x) \in \mathbb{R}$. A value of $S(t,x)$ near to zero indicates that the material is in the matrix phase at the point $x \in \Omega$ at time t, a value near to one indicates that the material is in the second phase. The other unknowns are the displacement $u(t,x) \in \mathbb{R}^3$ of the material point x at time t and the Cauchy stress tensor $T(t,x) \in \mathcal{S}^3$, where $\mathcal{S}^3$ denotes the set of symmetric 3×3-matrices. In what follows, ∇_x and Δ_x are, respectively, the gradient and Laplace operators. We use $\nabla_x u$ to denote the 3×3-matrix of first order derivatives of u, the deformation gradient, $(\nabla_x u)^T$ to denote the transposed matrix. And

$$\varepsilon(\nabla_x u) = \frac{1}{2}\left(\nabla_x u + (\nabla_x u)^T\right)$$

is the strain tensor. S_t denotes the partial derivative of S with respect to t, and

$$|\nabla_x S| = \Big(\sum_{i=1}^{3} |\partial_{x_i} S|^2\Big)^{\frac{1}{2}}.$$

Given are $b : [0,\infty)\times\Omega \to \mathbb{R}^3$, the volume force, $\bar{\varepsilon} \in \mathcal{S}^3$, a given matrix which is called the misfit strain, and $D : \mathcal{S}^3 \to \mathcal{S}^3$, the elasticity tensor that is chosen as a linear, symmetric, positive definite mapping. Finally, we choose the following free energy function ψ

$$\psi(\varepsilon, S) = \frac{1}{2}\big(D(\varepsilon - \bar{\varepsilon}S)\big)\cdot(\varepsilon - \bar{\varepsilon}S) + \hat{\psi}(S), \tag{1.1}$$

here ψ_S denotes the partial derivative of ψ, the function $\hat{\psi} \in C^2(\mathbb{R}, [0,\infty))$ is chosen as a double-well potential with two minima at $S = 0$ and $S = 1$, and one maximum in-between. The scalar product of two matrices is $A\cdot B = \sum a_{ij}b_{ij}$. And $c > 0$ is a constant and ν is a small positive constant.

Then the quasi-static equations for the unknown (u, T, S) read

$$-\operatorname{div}_x T = b, \tag{1.2}$$

$$T = D\big(\varepsilon(\nabla_x u) - \bar{\varepsilon}S\big), \tag{1.3}$$

$$S_t = -c\Big(\psi_S(\varepsilon(\nabla_x u), S) - \nu\Delta_x S\Big)|\nabla_x S|. \tag{1.4}$$

To derive this model, we choose a total free energy $\Psi(t) = \int_\Omega \psi^*(\varepsilon, S, \nabla_x S)dx$ with the density

$$\psi^*(\varepsilon, S, \nabla_x S) = \psi(\varepsilon, S) + \frac{\nu}{2}|\nabla_x S|^2,$$

and select a flux function

$$q = q(u_t, T, S_t, \nabla_x S) = Tu_t + \nu S_t \nabla_x S.$$

By straightforward computations we can find easily that if Eqs. (1.2)–(1.4) are satisfied, then there holds

$$\frac{\partial}{\partial t}\psi^* + \operatorname{div}_x q \le b\cdot u_t$$

in the sense of distribution. Namely, the validness of the second law of thermodynamics is guaranteed. For the details, we refer to Alber and Zhu [11,13].

Due to that the principal part in Eq. (1.4) degenerates as the gradient of S vanishes, and is not in divergence form in the multi-dimensional case, the investigation of the well-posedness of model (1.2)–(1.4) is thus difficult.