Nonlinear Analysis: Real World Applications 39 (2018) 77–92

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Global existence and blow-up of positive solutions of a parabolic problem with free boundaries

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ARTICLE INFO

Article history:
Received 28 November 2016
Received in revised form 10 June 2017
Accepted 16 June 2017
Available online 13 July 2017

Keywords:
Free boundary
Blow up
Fast solution
Slow solution

ABSTRACT

The purpose of this paper is to investigate a parabolic problem with double free boundaries. By the contraction mapping theorem, we establish the local existence and uniqueness of positive solutions. We present some sufficient conditions with respect to blow-up in finite time of the solution, and the existence of the fast solution and the slow solution, respectively. Finally, we obtain a trichotomy conclusion by considering the size of parameter σ.

1. Introduction

Consider a substance which is heat-diffusive and chemically reactive in its liquid phase, and neutral in its solid phase. Assume that the (one-dimensional) liquid is surrounded by the solid at melting temperature 0 at both ends. When an integral source term is considered, it is then lead to the following one-phase Stefan problem:

$$\begin{cases} u_t - du_{xx} = au^p \displaystyle\int_{g(t)}^{h(t)} u(t,x)^q \mathrm{d}x, & t>0, \quad g(t)<x<h(t), \\ u(t,g(t))=0, \quad g'(t) = -\mu u_x(t,g(t)), & t>0, \\ u(t,h(t))=0, \quad h'(t) = -\mu u_x(t,h(t)), & t>0, \\ g(0)=-h_0, \quad h(0)=h_0, \quad u(0,x)=u_0(x), & -h_0<x<h_0, \end{cases} \tag{1.1}$$

where both $x=g(t)$ and $x=h(t)$ are moving boundaries to be determined, $h_0>0, p>1, \mu>0$, $d>0$ and $q \geq 0$, and $au^p \int_{g(t)}^{h(t)} u(t,x)^q \mathrm{d}x$ is the integral source. In this paper, we always assume that the initial

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http://dx.doi.org/10.1016/j.nonrwa.2017.06.007

function $u_0(x)$ is a positive function satisfying

$$\begin{cases} u_0 \in C^2([-h_0, h_0]), \quad u_0 > 0, \\ u_0(-h_0) = u_0(h_0) = 0. \end{cases} \tag{1.2}$$

The free boundaries conditions, $g'(t) = -\mu u_x(t, g(t))$ and $h'(t) = -\mu u_x(t, h(t))$, are special cases of the well-known Stefan conditions, which have been used in the modeling of a number of applied problems. For example, they were used to describe the melting of ice in contact with water [1], in the modeling of oxygen in the muscle [2], and in wound healing [3]. There is a vast literature on the Stefan problem, and some important recent theoretical advances can be found in [4].

Recently, Li and Xie [5] considered the following problem with homogeneous Dirichlet boundary condition and positive initial value $u_0(x)$:

$$u_t - \Delta u = au^p \int_0^t u(t, x)^q ds, \ t > 0, \ x \in \Omega, \tag{1.3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $p > 0$ and $q \geq 0$. They proved the local existence and uniqueness of a positive classical solution, and obtained that the solution either exists globally or blows up in finite time by utilizing sub–supersolution techniques. Furthermore, they established the blow-up rate.

Ghidouche, Souplet and Tarzia [6] considered the following Stefan problem

$$\begin{cases} u_t - u_{xx} = u^p, & 0 < t < T, \ 0 < x < s(t), \\ u(0, x) = u_0(x) \geq 0, & 0 < x < s_0, \\ u(t, s(t)) = u_x(t, 0) = 0, & 0 < t < T, \\ s'(t) = -\mu u_x(t, s(t)), & t > 0, \end{cases} \tag{1.4}$$

where $p > 1$ and $s(0) = s_0 > 0$. They exhibited an energy condition, involving the initial data, under which the solution blows up in finite time in L^∞ norm. They also proved that all global solutions are bounded and decay uniformly to 0. Specifically, suppose that the maximal time $T^* = \infty$ holds, then one of the following two possibilities holds:

(i) Fast solution: $s_\infty := \lim_{t\to\infty} s(t) < \infty$ and there exist $C, \alpha > 0$ such that $|u|_\infty \leq Ce^{-\alpha t}$ for $t \geq 0$;

(ii) Slow solution: $s_\infty = \infty$ and $\lim_{t\to\infty} |u|_\infty = 0$. Moreover, $\liminf_{t\to\infty} t^{4/(3(p-1))} |u|_\infty > 0$.

Fila and Souplet [7,8] considered problem (1.4), they proved that there exist global solutions with slow decay.

Lately, Zhou, Bao and Lin [9] are concerned with a double free boundaries problem for the heat equation with a localized nonlinear reaction term

$$\begin{cases} u_t - du_{xx} = u^p(t, 0), & 0 < t < T, \ g(t) < x < h(t), \\ u(t, g(t)) = 0, \quad g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ u(t, h(t)) = 0, \quad h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ g(0) = -h_0, \quad h(0) = h_0, \quad u(0, x) = u_0(x), & -h_0 \leq x \leq h_0, \end{cases} \tag{1.5}$$

where $p > 1$, $d > 0$ and $\mu > 0$. The local existence and uniqueness of the solution of problem (1.5) were given by applying the contraction mapping theorem. And they present some conditions such that the solution blows up in finite time. Furthermore, they discussed the fast solution and slow solution of problem (1.5). For the spreading and vanishing dichotomy of reaction–diffusion–advection models, one can see [10–12]. In [13], Zhou and Lin studied the following problem

$$\begin{cases} u_t - du_{xx} = a \displaystyle\int_{g(t)}^{h(t)} u(t, x)^q dx, & t > 0, \ g(t) < x < h(t), \\ u(t, g(t)) = 0, \quad g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ u(t, h(t)) = 0, \quad h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ g(0) = -h_0, \quad h(0) = h_0, \quad u(0, x) = u_0(x), & -h_0 \leq x \leq h_0, \end{cases} \tag{1.6}$$