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# Dirichlet problem of a delay differential equation with spatial non-locality on a half plane

Wenjie Hu[a,b,*], Yueliang Duan[b], Yinggao Zhou[b]

[a] *The Journal House, Hunan Normal University, Changsha, Hunan 410081, PR China*
[b] *School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, PR China*

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ABSTRACT

In this paper, we are concerned with modeling and analyzing the dynamics for a two-stage species that lives on a half plane. We first derive a spatially nonlocal and temporally delayed differential equation that describes the mature population on a semi-infinite environment with a homogeneous Dirichlet condition. For the derived model, we are able to show that the solutions induce a $k$-set contraction semiflow with respect to the compact open topology on a bounded positive invariant set attracting every solution of the equation. To describe the global dynamics, we first establish a priori estimate for nontrivial solutions after exploring the delicate asymptotic properties of the nonlocal delayed effect, which enables us to show the repellency of the trivial equilibrium. Using the estimate, $k$-set contracting property as well as Schauder fixed point theorem, we then establish the existence of a positive spatially heterogeneous steady state. At last, we show global attractivity of the nontrivial steady state by employing dynamical system approaches.



## 1. Introduction

In recent years, there has been increasing interest in physiologically structured population dynamics (Metz and Diekmann [1] and Zhao [2]). When modeling the population dynamics of Nicholson's blowflies whose mature individuals diffuse that habitats in a bounded domain $\Omega$, Yang an So [3] considered the following spatially local equation by directly incorporating a spatial diffusion term $d\Delta u$ into the Nicholson's blowflies equation [4]

$$\frac{\partial u(t,x)}{\partial t} = d\Delta u(t,x) - \delta u(t,x) + pu(t-\tau)e^{-au(t-\tau)}, \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.1}$$

where $u(t,x)$ is the population of the adult flies at time $t$, $d$ is diffusion rate, $\delta$ is the per capita daily death rate, $p$ is the maximum per capita egg production rate, $1/a$ is the size at which the fly population

* Corresponding author at: The Journal House, Hunan Normal University, Changsha, Hunan 410081, PR China.
*E-mail address:* wenjhu@csu.edu.cn (W. Hu).

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reproduces at its maximum rate and $\tau$ is the maturation time. They studied the threshold dynamics of (1.1) under Neumann boundary value condition (NBVC) when $1 < p/\delta \le e$. Also under NBVC, Yi and Zou [5] proved the global attractivity of the unique positive constant equilibrium for (1.1) by combining a dynamical systems argument and some subtle inequalities in the case $e < p/\delta \le e^2$. In [6], the authors further obtained some very sharp results for delay independent global stability of steady states of spatially local model (1.1) by employing the idea of attracting intervals for solution semiflow (1.1). For the Dirichlet problem, So and Yang [7] obtained some results, by energy method, on the existence and stability of a positive steady of the spatially local model (1.1). By modifying some of the arguments in Yi and Zou [6], Yi et al. [8] investigated threshold dynamics of a general unimodal equation which includes (1.1) as a special case.

In order to describe the population dynamics of the species whose both adult and juvenile diffuse and lives on bounded domains, Xu and Zhao [9] proposed the following model

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = d\Delta u(t,x) - g(u(t,x)) + \displaystyle\int_{\Omega} \Gamma(\eta(\tau),x,y)\mathcal{F}(\tau)f(u(t-\tau,y))\mathrm{d}y, & t>0,\ x\in\Omega, \\ Bu(t,x) = 0, & t>0,\ x\in\partial\Omega. \end{cases} \tag{1.2}$$

Here $u(t,x)$ represents the total matured population at time $t$ and location $x$, $d$ is the diffusion rate of the matured population, $\tau$ is the length of the juvenile period, $f(u)$ and $g(u)$ are the birth rate and mortality rate of the matured population, $\Omega$ is a bounded and open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. $B$ is the boundary operator and $\Gamma$ is the Green function associated with the partial differential operator $\Delta$ and boundary condition $Bu = 0$. The other two indirect parameters $\eta(\tau)$ and $\mathcal{F}(\tau)$ are defined by $\eta(\tau) = \int_0^\tau d_j(s)\mathrm{d}s$ and $\mathcal{F}(\tau) = e^{-\int_0^\tau \mu_j(s)\mathrm{d}s}$, where $d_j(s)$, $\mu_j(s)$, $s\in[0,\tau]$ are the age dependent death rate and diffusion rate of the immature population of species. In [9], the authors obtained some results on threshold dynamics and the uniqueness and global attractiveness of a positive steady state of (1.2) by assuming sublinearity on the nonlocal nonlinear term. In [10], Liang et al. derived the specific kernels $\Gamma$ in model (1.2) under Dirichlet boundary value condition (DBVC), NBVC and mixed boundary value condition respectively and analyzed the numerical solutions as well. In the non-monotone case, describing the dynamics of (1.2) turns out to be more difficult. Nevertheless, some advances have been made. For instance, when NBVC is imposed, by applying a fluctuation method, Zhao [11] has recently established the global attractiveness of the positive steady state of (1.1). For the Dirichlet problem, Yi and Zou [12] established threshold results and global attractiveness of the trivial steady state as well as the existence, uniqueness and global attractiveness of a positive steady state for spatially nonlocal model (1.2) by combining comparison technique and a dynamical approach.

Based on the interaction of intrinsic dynamics (birth and death) and the spatial diffusion in a structured population, So et al. [13] derived a spatially nonlocal and temporally delayed model described by the following reaction–diffusion equation

$$\frac{\partial u}{\partial t}(t,x) = D_m\Delta u(t,x) - d_m u(t,x) + \epsilon\int_{\mathbb{R}} f_\alpha(x-y)b(u(t-\tau,y))\mathrm{d}y, \quad (t,x)\in(0,\infty)\times\mathbb{R}. \tag{1.3}$$

Here $u(t,x)$ represents the total matured population at time $t$ and location $x$, $D_m$ and $d_m$ are the diffusion rate and death rate for the matured population, $\tau$ is the maturation time for the species, the other two indirect parameters $\varepsilon$ and $\alpha$ are defined by $\varepsilon = e^{-\int_0^\tau d_I(a)\mathrm{d}a}$, $\alpha = \int_0^\tau D_I(a)\mathrm{d}a$, where $d_I(a)$, $D_I(a)$, $a\in[0,\tau]$ are the age dependent death rate and diffusion rate of the immature population of the species, $b(u)$ is the birth function and the kernel $f_\alpha(x)$ parameterized by $\alpha$ is given by $f_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{x^2}{4\alpha}}$. When mature individuals of the species diffuse but their immature individuals do not, such as birds, by letting $\alpha\to 0$, Eq. (1.3) is reduced to the following spatially local model

$$\frac{\partial u(t,x)}{\partial t} = D_m\Delta u(t,x) - d_m u(t,x) + \epsilon b(u(t-\tau,x)), \quad (t,x)\in(0,\infty)\times\mathbb{R}. \tag{1.4}$$