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The generalized polynomial Moon–Rand system

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ABSTRACT

The Moon–Rand systems, developed to model control flexible space structures, are systems of differential equations in $\mathbb{R}^3$ of the form

$$\dot{u} = v, \qquad \dot{v} = -u - uw, \qquad \dot{w} = -\lambda w + f(u, v).$$

We give a partially positive answer to a recently conjecture for a special class of such systems, called the generalized polynomial Moon–Rand systems in the case when $\lambda \in (0, \infty)$ and f is either a homogeneous cubic, quartic, quintic or sextic polynomial.

1. Introduction and statement of the main results

The Moon–Rand systems are systems of differential equations on $\mathbb{R}^3$ with polynomial or rational right hand sides having an isolated singularity at the origin at which the linear part has one negative and one pair of purely imaginary eigenvalues for all choices of the parameters. These systems were developed to model control of flexible space structures. More specifically, Moon and Rand [1] (see also Exercise 5 of Section 5.5 of [2]) introduced the following system of differential equations

$$\dot{u} = v, \qquad \dot{v} = -u - uw, \qquad \dot{w} = -\lambda w + f(u, v), \tag{1}$$

which we shall call the *Moon–Rand system* and where $f(u, v) = c_{20}u^2 + c_{11}uv + c_{02}v^2$ or $f(u, v) = c_{11}uv/(1 + \eta u^2)$. Here λ, η, c_{20}, c_{11} and c_{02} are real numbers and $\lambda > 0$ and $\eta > 0$. The authors showed that in the polynomial case the origin is asymptotically stable for the flow restricted to the center manifold if $2c_{20} - 2c_{02} - \lambda c_{11} < 0$. Recently in [3] the authors have given a complete analysis of the flow restricted to a neighborhood of the origin in any center manifold for arbitrary values of η and strictly positive values of λ so that the singularity at the origin becomes isolated.

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In [3] is given a natural generalization of system (1) in the polynomial case. More precisely, the authors in [3] consider that

$$f(u,v) = \sum_{j+k=n} c_{jk} u^j v^k,$$

and the resulting systems with such f are called the *generalized polynomial Moon–Rand systems*.

In this paper, we study a conjecture posed in [3] related to these generalized polynomial Moon–Rand systems. More concretely, in [3] the authors conjectured that in the case of f being a homogeneous polynomial of degree n, called f_n, then system (1) has a center on the local center manifold at the origin if and only if

$$f_n(u,v) = c\,u^n + \frac{n}{\lambda} c\,u^{n-1} v.$$

They proved this for the case $\lambda \in (0,\infty)$ and $f = f_2$, i.e., f being a homogeneous polynomial of degree two and also in the case $\lambda = 1$ and $f = f_3$, that is, f being a homogeneous polynomial of degree three. In [4] the author proves the conjecture for the case $\lambda = 2$ and $f = f_3$ a homogeneous polynomial of degree three. Due to the enormous computation restrictions the authors in [3,4] state that they do not consider the general case of $\lambda \in (0,\infty)$ and $f = f_3$. We recall that the case $f = f_3$ is computationally much harder than the case $f = f_2$ due to the presence of an extra parameter. Moreover, the parameter λ appears in the denominator of the focus quantities and growing λ brings computational difficulties. Finally, we want to mention that going from a fixed value of λ to considering it as a parameter provides again enormous computation difficulties. This is the main issue so that the authors either in [3] or in [4] could not consider the general case. In this paper using another smart approach we are able to partially solve the conjecture for the case $\lambda \in (0,\infty)$ and $f = f_3$, $f = f_4$, $f = f_5$ or $f = f_6$. This gives the intuition that the conjecture must be right. More concretely, we prove the following results. We recall that a monodromic system with a nondegenerate singular point has a center if and only if there exists a locally analytic first integral defined in a neighborhood of it.

We define *Property A*: If system (1) with $\lambda \in (0,\infty)$ and f a homogeneous polynomial of degree n has a center on the center manifold at the origin, then either $f_n = c_1 u^n + \frac{n}{\lambda} c_1 u^{n-1} v$, or it has a local analytic first integral in the variables x and y and continuous in the parameters of the system in a neighborhood of zero in the parameter space formed by the free parameters once we have a fixed center.

Theorem 1. *System (1) with $\lambda \in (0,\infty)$ and f a homogeneous polynomial of degree three satisfies Property A if and only if $f = c_1 u^3 + \frac{3}{\lambda} c_1 u^2 v$.*

Theorem 1 is proved in Section 3. Theorem 1 is also true in the case of a homogeneous polynomial of degrees four, five or six. The proof in these cases is analogous to the proof when we have a homogeneous polynomial of degree three and we will not do it to save the reader of repetitive developments. Thus, we state the results without proof. The approach used in this work is also used in [5,6].

Theorem 2. *System (1) with $\lambda \in (0,\infty)$ and f a homogeneous polynomial of degree n for $n = 4, 5$ and 6 satisfies Property A if and only if $f = c_1 u^n + \frac{n}{\lambda} c_1 u^{n-1} v$.*

Note that Theorems 1 and 2 prove the conjecture stated in [3] for the case of f being a cubic, quartic, quintic or sextic homogeneous polynomial and $\lambda \in (0,\infty)$ with the restriction that the system satisfies Property A. We believe that the same method of proof works for the case $\lambda \in (0,\infty)$ and $f = f_n$ a homogeneous polynomial of degree $n \geq 7$ but it is computationally too hard and we cannot treat it now with our computational facilities.

In the next section we provide general techniques and definitions necessary for the proof of our main theorem. We recall that in [3] the authors provide the Mathematica code for automatic computation of the coefficients of the lowest order terms in the expansion of the center manifold in a neighborhood of the origin and of the first focus quantities in an abbreviated form due to its length.