Nonlinear Analysis: Real World Applications 39 (2018) 451–463

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Bifurcation of equilibrium forms of an elastic rod on a two-parameter Winkler foundation

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ARTICLE INFO

Article history:
Received 3 February 2017
Accepted 15 July 2017
Available online 18 August 2017

Keywords:
Bifurcation
Buckling
Winkler foundation

ABSTRACT

We consider two-parameter bifurcation of equilibrium states of an elastic rod on a deformable foundation. Our main theorem shows that bifurcation occurs if and only if the linearization of our problem has nontrivial solutions. In fact our proof, based on the concept of the Brouwer degree, gives more, namely that from each bifurcation point there branches off a continuum of solutions.

1. Introduction

Bifurcation theory is one of the most powerful tools in studying deformations of elastic beams, plates and shells. Numerous works have been devoted to the study of bifurcation in elasticity theory (see for instance [1,2] and the references therein).

A familiar example from beam theory is the problem of stability of an isotropic elastic rod lying on a deformable foundation which is being compressed by forces at the ends (see Fig. 1). For small forces the rod maintains its shape, however, as the forces increase they reach a first critical value beyond which the rod may buckle.

In this work, we consider mixed boundary conditions which are as follows. The beam is free at the left end, and so it may move as in Fig. 2. However, we require the shear force at the left end to vanish. At the right end, we assume the beam to be simply supported.

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http://dx.doi.org/10.1016/j.nonrwa.2017.07.008
1468-1218/

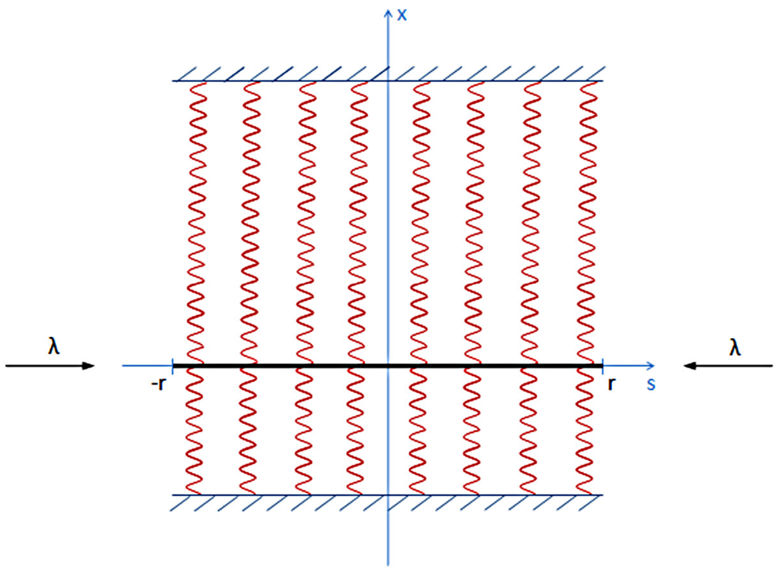

Fig. 1. An elastic beam on an elastic foundation.

As we will show later, equilibrium forms of the rod under these boundary conditions satisfy the boundary value problem

$$\begin{cases} x^{(4)} + \alpha x'' + \beta x - f(x, x', \dots, x^{(4)}) = 0, & \text{in } [-r, r], \\ x'(-r) = x'''(-r) = 0, \\ x(r) = x''(r) = 0, \end{cases} \tag{1}$$

where α is a parameter of the compressive force, β is a parameter of the elastic foundation, and f is a nonlinear term which we define in (9) below. It follows from the definition of f that for small forces the only solution of (1) is the trivial one, i.e. $x_0(s) = 0$, $s \in [-r, r]$, which corresponds to the straight rod in our bifurcation model.

However, as the forces increase the rod may buckle and it is desirable to know for which positive parameter values (α, β) this might happen.

In order to answer this question, we associate with (1) the linear boundary value problem

$$\begin{cases} x^{(4)} + \alpha x'' + \beta x = 0, & \text{in } [-r, r], \\ x'(-r) = x'''(-r) = 0, \\ x(r) = x''(r) = 0 \end{cases} \tag{2}$$

and we denote by $N(\alpha, \beta)$ its space of solutions.

The main theorem of this paper shows that a necessary and sufficient condition for bifurcation, and so for the possibility of a buckling of the rod, is that $\dim N(\alpha, \beta) \neq 0$.

Let us point out that a similar model was investigated by A. Borisovich, Yu. Morozov and Cz. Szymczak in [3], where the authors assumed that the rod is simply supported at both ends. They proved the existence of simple bifurcation points (meaning that $\dim N(\alpha, \beta) = 1$) by applying a variational version of the Crandall–Rabinowitz theorem (compare Theorem 3.4 below). Later, in [4], A. Borisovich and J. Dymkowska showed a corresponding result under our boundary conditions, however, to the best of our knowledge the existence of multiple bifurcation points in the solution set of (1) is new. Note that here we prove even more, namely the existence of multiple branching points.

Finally, let us mention that other models for buckling are described for example in [2,5–10].

Our paper is composed of three sections. In Section 2 we derive the equation of equilibrium forms of the rod and state our main theorem. Section 3 is devoted to the proof of this result.