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Thresholds for hanger slackening and cable shortening in the Melan equation for suspension bridges

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ABSTRACT

The Melan equation for suspension bridges is derived by assuming small displacements of the deck and inextensible hangers. We determine the thresholds for the validity of the Melan equation when the hangers slacken, thereby violating the inextensibility assumption. To this end, we preliminarily study the possible shortening of the cables: it turns out that there is a striking difference between even and odd vibrating modes since the former never shorten the cables. These problems are studied both on beams and plates.

1. Introduction

In 1888, the Austrian engineer Josef Melan [1] introduced the so-called deflection theory and applied it to derive the differential equation governing a suspension bridge, modeled as a combination of a string (the sustaining cable) and a beam (the deck), see Fig. 1. The beam and the string are connected through hangers. Since the spacing between hangers is usually small relative to the span, the set of the hangers is considered as a continuous membrane connecting the cable and the deck.

Let us quickly outline how the Melan equation is derived; we follow here [2, VII.1]. We denote by
L the length of the beam at rest (the distance between towers) and $x \in (0, L)$ the position on the beam;
$p = p(x)$ the live load and $-q < 0$ the dead load per unit length applied to the beam;
$g = g(x)$ the displacement of the cable due to the dead load $-q$;
L_c the length of the cable subject to the dead load $-q$;
A the cross-sectional area of the cable and E_c its modulus of elasticity;
H the horizontal tension in the cable, when subject to the dead load $-q$ only;
EI the flexural rigidity of the beam;

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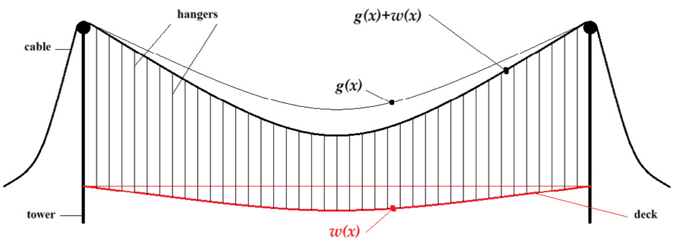

Fig. 1. Beam (w) sustained by a cable (g) through parallel hangers.

$w = w(x)$ the displacement of the beam due to the live load p;
$h = h(w)$ the additional tension in the cable produced by the live load p.

When the system is only subject to the action of dead loads, the cable is in position $g(x)$ while the unloaded beam is in the horizontal position $w \equiv 0$, see Fig. 1. The cable is adjusted in such a way that it carries its own weight, the weight of the hangers and the weight of the deck (beam) without producing a bending moment in the beam, so that all additional deformations of the cable and the beam due to live loads are small. The cable is considered as a perfectly flexible string subject to vertical dead and live loads. The string is subject to a downwards vertical constant dead load $-q$ and the horizontal component $H > 0$ of the tension remains constant. If the mass of the cable is neglected, then the dead load is distributed per horizontal unit. The resulting equation simply reads $Hg''(x) = q$ (see [2, (1.3), VII]) so that the cable takes the form of a parabola with a ∪-shaped graph. If the endpoints of the string (top of the towers) are at the same level $\gamma > 0$ (as in suspension bridges, see again Fig. 1), then the solution g and the length L_c of the cable are given by:

$$g(x) = \gamma + \frac{q}{2H} x(x-L)\,, \quad g'(x) = \frac{q}{H}\left(x - \frac{L}{2}\right)\,, \quad g''(x) = \frac{q}{H}\,, \quad \forall x \in (0, L), \tag{1}$$

$$L_c = \int_0^L \sqrt{1+g'(x)^2}\, dx. \tag{2}$$

The elastic deformation of the hangers is usually neglected, so that the function w describes both the displacements of the beam and of the cable from its equilibrium position g. This classical assumption is justified by precise studies on linearized models, see e.g. [3]. When the live load p is added, a certain amount p_1 of p is carried by the cable whereas the remaining part $p - p_1$ is carried by the bending stiffness of the beam. In this case, it is well-known [1,2] that the equation for the displacement w of the beam is

$$EI\, w''''(x) = p(x) - p_1(x) \qquad \forall x \in (0, L)\,. \tag{3}$$

At the same time, the horizontal tension of the cable is increased to $H + h(w)$ and the deflection w is added to the displacement g. Hence, according to (1), the equation which takes into account these conditions reads

$$\big(H + h(w)\big)\big(g''(x) + w''(x)\big) = q - p_1(x) \qquad \forall x \in (0, L)\,. \tag{4}$$

Then, by combining (1)–(3)–(4), we obtain

$$EI\, w''''(x) - \big(H + h(w)\big)\, w''(x) - \frac{q}{H}\, h(w) = p(x) \qquad \forall x \in (0, L)\,, \tag{5}$$

which is known in literature as the **Melan equation** [1, p. 77]. The beam representing the bridge is hinged at its endpoints, which means that the boundary conditions to be associated to (5) are

$$w(0) = w(L) = w''(0) = w''(L) = 0\,. \tag{6}$$