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Equivariant bifurcations in a non-local model of ferromagnetic materials

Andréia S. Coutinho[a], Antônio L. Pereira[b,*]

[a] *Universidade Federal de Lavras-MG, Brazil*

[b] *Instituto de Matemática e Estatística-Universidade de São Paulo, Rua do Matão, 1010, Cidade Universitária, 05508-090, São Paulo-SP, Brazil*

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ABSTRACT

In this work we study the bifurcations from the trivial equilibrium of the equation

$$\frac{\partial u}{\partial t}(x,t) = -u(x,t) + \tanh(\beta(J * u)(x,t)),$$

in the space of 2τ periodic functions. This is accomplished with the help of the *equivariant branching lemma*, which allows us to take into account the symmetries present in the model. We show that the phenomenon of 'spontaneous symmetry-breaking' occurs here, that is, the bifurcating solutions are less symmetric than the trivial one. We also prove that, under certain conditions, these equilibria can be globally continued.

1. Introduction

We consider here the non-local evolution equation

$$\frac{\partial u(r,t)}{\partial t} = -u(r,t) + \tanh\left(\beta J * u(r,t)\right), \tag{1}$$

where $u(r,t)$ is a real function on $\mathbb{R} \times \mathbb{R}_+$, β is a positive constant and $J \in C^1(\mathbb{R})$ is a nonnegative even function with compact support and integral equal to 1. The $*$ above denotes convolution product, namely:

$$(J * m)(x) = \int_{\mathbb{R}} J(x-y)m(y)dy.$$

Non-local convolution models appear in the modelling of many phenomena, including population dynamics [1–3] and neuronal activity [4–11].

* Corresponding author.
E-mail addresses: andreia@dex.ufla.br (A.S. Coutinho), alpereir@ime.usp.br (A.L. Pereira).

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In particular, Eq. (1) arises as a continuum limit of one-dimensional Ising spin systems with Glauber dynamics and Kac potential (see [12] and references therein); u represents then a magnetization density and β^{-1} the temperature of the system. Though 'local', e.g. reaction–diffusion, differential equations have also been used in this context, Eq. (1) seems to be the 'right' one to use, especially if one is interested in phase transitions and metastability properties (see [13]).

It is not difficult to obtain well posedness of the problem (1) in various function spaces since the right-hand-side of (1) usually defines a global Lipschitz map. On the other hand, the investigation of qualitative properties of the associated flow seems to be harder than with the corresponding local model. To begin with, the equilibria are given by the solutions of a nonlinear integral equation for which many methods used to analyse, for example, the boundary value problems that appear in the case of semilinear parabolic problems are not available. Furthermore, as it is shown in [14], non-locality can give rise to complicated dynamics even in the case of scalar parabolic equations.

In the last years several works dedicated to the analysis of (1) appeared in the literature. In [15,16], the existence and uniqueness (modulo translations) of a travelling front connecting the equilibria m_β^- and m_β^+ is proved. In the case $h = 0$ the existence of a 'standing' wave as well as its stability properties are analysed in [13,17]. In this case, many equilibria periodic in x also exist, as shown in [18,19]. The existence of a non-homogeneous stationary solution referred to as the 'bump' or 'critical droplet' in the literature, was proved in [20] for h 'sufficiently close' to 1. Another proof, which is simpler and does not require the above restriction in h is given in [21]. In the same work, the existence of a global compact attractor in $L^2(\mathbb{R})$ with a convenient weighted measure is obtained.

We consider here Eq. (1) restricted to the subspace $\mathbb{P}_{2\tau}$ of 2τ—periodic functions (with support of J contained in $[-\tau, \tau]$). The existence and continuity of global attractors with respect to parameters were proved for a slightly more general class of equations in [22,23]. Our aim is to investigate the appearance of (families of) solutions bifurcating from the trivial solution of (1) (or, more precisely, its restriction to $\mathbb{P}_{2\tau}$). This equation has a $\mathbb{O}(2) \times \mathbb{Z}_2$-type symmetry and, as it is well-known, after the work of many researchers in the late seventies, this has profound consequences on the bifurcation of its equilibria (see, for example, [24,25]). In particular, the trivial solution has full G-symmetry, but as we shall see, the bifurcating solutions are less symmetric, that is, they are symmetric under a proper subgroup of G, a phenomenon known as *spontaneous symmetry breaking* in the literature. Similar situations, for the case of semilinear elliptic equations, have been discussed by many authors, including [24–28]. The main tool used is the Equivariant Branching Lemma proved independently in [29,30] in a version given in [24]. This result may be seen as an equivariant version of the famous Crandall–Rabinowitz bifurcation theorem, giving sufficient conditions for bifurcation to occur in the presence of symmetries and information on the type of symmetry enjoyed by the bifurcating solutions.

We also prove that, if the support of J is sufficiently small, the equilibria obtained by the Equivariant Branching Theorem can be globally continued.

This paper is organized as follows: in Section 2 we review some concepts and results from the representation theory of compact groups needed in the statement of the Equivariant Branching Theorem. In Section 3, after establishing some spectral properties of the convolution operator, we apply the Equivariant Branching Theorem to obtain our local bifurcation results. We also study the stability properties of these solutions. Finally, in Section 4 we prove a result of global continuation for the bifurcated solutions.

2. Preliminaries

2.1. Some concepts from group representation theory

We first recall some definitions from the representation theory of compact groups. For more details and proofs we refer to [25] or [24,31].