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# Ground states for pseudo-relativistic Choquard equations

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We investigate the existence of positive ground states for pseudo-relativistic nonlinear Choquard equations. Our results are based on Nehari manifold technique and rearrangement methods. Furthermore, we also obtain the nonrelativistic limit of ground states in $H^1(\mathbb{R}^N)$ space.



## 1. Introduction and main results

Consider the following pseudo-relativistic Choquard equations

$$i\frac{\partial\psi}{\partial t} = \sqrt{-c^2\Delta + m^2c^4}\psi - mc^2\psi - (|x|^{\alpha-N} * |\psi|^p)|\psi|^{p-2}\psi, \quad x \in \mathbb{R}^N, t \in \mathbb{R}, \tag{1.1}$$

where $m > 0$ denotes the particle mass, $c > 0$ is the speed of light, $N \geq 2$, $\alpha \in (0, N)$ and $p > 1$, $\psi(t, x)$ is a complex-valued wave field, and the symbol $*$ stands for convolution on $\mathbb{R}^N$.

Recently, the study of the problem (1.1) has attracted the attentions of many authors. As $c = 1$, $p = 2$, $\alpha = 2$ and $N = 3$, the problem (1.1) is known to describe the dynamics of pseudo-relativistic boson stars in the meanfield limit. Fröhlich, Jonsson and Lenzmann [1], Lenzmann [2] proved the existence and uniqueness of solitary wave solution. For more details and applications, we refer to Refs. [3–5].

Due to the focusing nature of the nonlinearity in the problem (1.1), there exist solitary solutions given by

$$\psi(x, t) = e^{i\omega t}u(x), \quad \omega \in \mathbb{R}, \tag{1.2}$$

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where $u(x) \in H^{\frac{1}{2}}(\mathbb{R}^N)$. So we obtain the following nonlinear nonlocal problem

$$\sqrt{-c^2\Delta + m^2c^4}u(x) - mc^2u(x) + \omega u(x) = (|x|^{\alpha-N} * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.3}$$

In this paper, we investigate the ground state solution for the problem (1.3) and obtain the relation between the ground state solution and $L^2$-constrained minimum solution. On the other hand, when we are able to pass to the limit in the problem (1.3) as $c \to \infty$, and obtain a one parameter family of solution $\{u_c\}$ of the problem (1.3) converges, up to a subsequence, to a solution for relativistic version of the limit equations

$$-\frac{1}{2m}\Delta u(x) + \omega u(x) = (|x|^{\alpha-N} * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.4}$$

Now, we state our main theorems in this paper.

**Theorem 1.1.** *Let $m, \omega > 0$, $c \geq 1$ and $p$ satisfies $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-1}$, then there exists a ground state solution $u(x) \in H^{\frac{1}{2}}(\mathbb{R}^N)$ for the problem (1.3) which is positive, radially symmetric.*

**Theorem 1.2.** *Let $m, \omega > 0$, $c \geq 1$ and $p$ satisfies $1+\frac{\alpha}{N} < p < \frac{N+\alpha}{N-1}$, and $u_c$ be the positive radially symmetric ground states of the problem (1.3) which is obtained in Theorem 1.1. Then a subsequence $\{u_c\}$ converges to a unique positive radially symmetric solution $u_\infty$ of*

$$-\frac{1}{2m}\Delta u_\infty + \omega u_\infty = (|x|^{\alpha-N} * |u_\infty|^p)|u_\infty|^{p-2}u_\infty, \quad x \in \mathbb{R}^N,$$

*as $c \to \infty$ in the sense that*

$$\lim_{c\to\infty} \|u_c - u_\infty\|_{H^1(\mathbb{R}^N)} = 0. \tag{1.5}$$

The rest of the paper is organized as follows. In Section 2, we recall the definition and some basic properties of the function spaces, consider the extension problem which localizes the nonlocal problem (1.3). In Section 3, we obtain the existence results. In Section 4, we obtain uniform estimates of the ground state solution in $H^1(\mathbb{R}^N)$. In Section 5, we prove the nonrelativistic limit.

## 2. Functional settings and the extension problem

The fractional operator $\sqrt{-c^2\Delta + m^2c^4}$ is defined in the Fourier space by setting

$$\mathcal{F}\left(\sqrt{-c^2\Delta + m^2c^4}\right)u(k) = (c^2|k|^2 + m^2c^4)^{\frac{1}{2}}\mathcal{F}u(k) = (c^2|k|^2 + m^2c^4)^{\frac{1}{2}}|\hat{u}(k)|.$$

We employ the Sobolev space, $H^{\frac{1}{2}}(\mathbb{R}^N)$ of fractional order $\frac{1}{2}$ defined by

$$H^{\frac{1}{2}}(\mathbb{R}^N) = \{u \in S^1(\mathbb{R}^N) : (1-\Delta)^{\frac{1}{4}}u \in L^2(\mathbb{R}^N)\},$$

and equipped with the norm $\|u\|_{H^{\frac{1}{2}}(\mathbb{R}^N)} := \|(1-\Delta)^{\frac{1}{4}}u\|_{L^2(\mathbb{R}^N)}$.

For the problem (1.3), we define the functional $I : H^{\frac{1}{2}}(\mathbb{R}^N) \to \mathbb{R}$ as follows

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(\sqrt{c^2k^2 + m^2c^4} - mc^2\right)|\hat{u}(k)|^2 + \omega|\hat{u}(k)|^2 dk - \frac{1}{2p}\int_{\mathbb{R}^N}(|x|^{\alpha-N} * |u|^p)|u|^p dx. \tag{2.1}$$

So the critical point $u$ of $I$ solves the problem (1.3). The ground state solution $u$ of the problem (1.3) if it satisfies

$$I(u) = \min\left\{I(v) | v \in H^{\frac{1}{2}}(\mathbb{R}^N) \setminus \{0\}, I'(v) = 0\right\}.$$