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Degenerate Kirchhoff-type diffusion problems involving the fractional p-Laplacian

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ABSTRACT

In this paper we study the existence of a global solution for a diffusion problem of Kirchhoff type driven by a nonlocal integro-differential operator. As a particular case, we consider the following parabolic equation involving the fractional p-Laplacian:

$$\begin{cases} \partial_t u + [u]_{s,p}^{(\lambda-1)p}(-\Delta)_p^s u = |u|^{q-2}u, & \text{in } \Omega \times \mathbb{R}^+, \quad \partial_t u = \partial u/\partial t, \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}_0^+, \end{cases}$$

where $[u]_{s,p}$ is the Gagliardo p–seminorm of u, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $p < q < Np/(N - sp)$ with $1 < p < N/s$ and $s \in (0, 1)$, $1 \leq \lambda < N/(N - sp)$, $(-\Delta)_p^s$ is the fractional p-Laplacian. Under some appropriate assumptions, we obtain the existence of a global solution for the problem above by the Galerkin method and potential well theory. It is worth pointing out that the main result covers the degenerate case, that is the coefficient of $(-\Delta)_p^s$ can vanish at zero.

1. Introduction

In [1], *Sattinger* introduced the potential well in order to prove the global existence of solutions for hyperbolic equations. Since then, many authors have studied the existence of solutions for evolution equations by potential well theory, see for example [2–9]. In [8], *Tsutsumi* studied the application of potential well

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to nonlinear parabolic equation. In [3], *Ikehata* and *Suzuki* treated stable and unstable sets for evolution equations of hyperbolic and parabolic equation by potential well method. In [4–7,9], *Liu* et al. applied potential well method to discuss the existence of solutions for semilinear wave equations, parabolic equations and double dispersion equations.

On the other hand, fractional and non-local problems are actively studied in recent years. Since the pioneering works of *Caffarelli* and *Silvestre* in [10], who introduced the s-harmonic extension to define the fractional Laplacian operator, many interesting results in the classical elliptic problems have been extended in the setting of the fractional Laplacian, one can see [11–15] and the references therein.

In this paper, we are interested in the following parabolic equations of Kirchhoff type involving the generalized fractional p-Laplacian:

$$\begin{cases} \partial_t u + M([u]^p_{s,p})\mathscr{L}^p_K u = f(u), & \text{in } \Omega\times\mathbb{R}^+, \quad \partial_t u = \partial u/\partial t,\\ u(x,0) = u_0(x), & \text{in } \Omega,\\ u(x,t) = 0, & \text{in } (\mathbb{R}^N\setminus\Omega)\times\mathbb{R}^+_0, \end{cases} \tag{1.1}$$

where $f(u) = |u|^{q-2}u$, p, q satisfy $p < q < p^*_s := Np/(N - sp)$ with $1 < p < N/s$, $M(t) = t^{\lambda-1}$ for $t \in \mathbb{R}^+_0$, $\lambda \in [1, p^*_s/p)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary and

$$\mathscr{L}^p_K\varphi(x) = 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} |\varphi(x)-\varphi(y)|^{p-2}[\varphi(x)-\varphi(y)]K(x-y)dy,$$

$$[\varphi]_{s,p} = \|\varphi\|_{W_0} = \left(\iint_Q |\varphi(x)-\varphi(y)|^p K(x-y)dxdy\right)^{\frac{1}{p}}$$

for any $\varphi \in C^\infty_0(\mathbb{R}^N)$, where $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ with radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^N$. The kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ satisfies the following assumption (K)

$$\begin{cases} mK \in L^1(\mathbb{R}^N), \quad \text{where } m(x) = \min\{|x|^p, 1\};\\ \text{there exists } K_0 > 0 \text{ such that } K(x) \geq K_0|x|^{-(N+ps)} \quad \text{for a.e. } x \in \mathbb{R}^N\setminus\{0\}. \end{cases}$$

A typical example for K is given by singular kernel $K(x) = |x|^{-(N+ps)}$. In this way, $\mathscr{L}^p_K\varphi(x) = (-\Delta)^s_p\varphi(x)$ for all $\varphi(x) \in C^\infty_0(\mathbb{R}^N)$, we refer to [16–19] for further details on the fractional Laplacian and the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$. In this case, $[\varphi]_{s,p}$ becomes the celebrated Gagliardo semi-norm.

To explain the motivation of problem (1.1), let us shortly introduce a prototype of nonlocal problem like (1.1) in $\mathbb{R}^+\times\mathbb{R}^N$. Indeed, nonlocal evolution equations of the form

$$\partial_t u(x,t) = \int_{\mathbb{R}^N}[u(y,t) - u(x,t)]\kappa(x-y)dy, \tag{1.2}$$

and its variants, have been recently widely used to model diffusion processes. More precisely, as stated by *Fife* in [20], if $u = u(x,t)$ is thought of as a density of population at the point x and time t and $\kappa(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} u(y,t)\kappa(x-y)dy$ is the rate at which individuals are arriving at position x from all other places and $-\int_{\mathbb{R}^N} u(x,t)\kappa(x-y)dy$ is the rate at which they are leaving location x to travel to all other sites. If we consider the effects of total population, then problem (1.2) becomes

$$\partial_t u(x,t) = M\left(\iint_{\mathbb{R}^{2N}}|u(x,t)-u(y,t)|^2\kappa(x-y)dxdy\right)\int_{\mathbb{R}^N}[u(y,t)-u(x,t)]\kappa(x-y)dy, \tag{1.3}$$