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Polynomial profits in renewable resources management

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ABSTRACT

A system of renewal equations on a graph provides a framework to describe the exploitation of a biological resource. In this context, we formulate an optimal control problem, prove the existence of an optimal control and ensure that the target cost function is polynomial in the control. In specific situations, further information about the form of this dependence is obtained. As a consequence, in some cases the optimal control is proved to be necessarily bang–bang, in other cases the computations necessary to find the optimal control are significantly reduced.

1. Introduction

A biological resource is grown to provide an economical profit. Up to a fixed age $\bar{a}$, this population consists of *juveniles* whose density $J(t,a)$ at time t and age a satisfies the usual renewal equation [1, Chapter 3]

$$\partial_t J + \partial_a \left(g_J(t,a)\, J\right) = d_J(t,a)\, J \qquad a \in [0,\bar{a}],$$

g_J and d_J being, respectively, the usual growth and mortality functions, see also [2–4]. For further structured population models, we refer for instance to [5–9].

At age $\bar{a}$, each individual of the J population is selected and directed either to the market to be sold or to provide new juveniles through reproduction. Correspondingly, we are thus lead to consider the S and the R populations whose evolution is described by the renewal equations

$$\begin{aligned} \partial_t S + \partial_a \left(g_S(t,a)\, S\right) &= d_S(t,a)\, S \\ \partial_t R + \partial_a \left(g_R(t,a)\, R\right) &= d_R(t,a)\, R \end{aligned} \qquad a \geq \bar{a},$$

with obvious meaning for the functions g_S, g_R, d_S, d_R. Here, the selection procedure is described by a parameter η, varying in $[0,1]$, which quantifies the percentage of the J population directed to the market,

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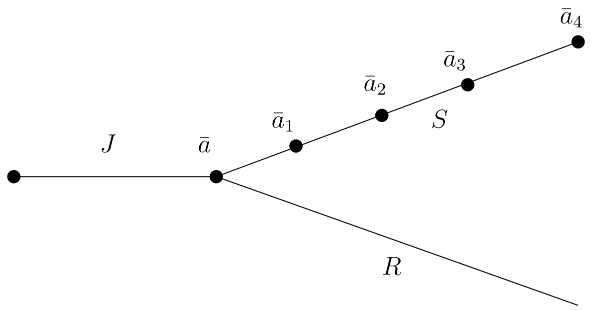

Fig. 1. The graph corresponding to the biological resource. At age $\bar a$, juveniles reach the adult stage and are selected. The part R is used for reproduction. Portions of the S population are sold at ages $\bar a_1, \ldots, \bar a_4$.

so that

$$\begin{aligned} g_S(t,\bar a)\, S(t,\bar a) &= \eta\, g_J(t,\bar a)\, J(t,\bar a) \\ g_R(t,\bar a)\, R(t,\bar a) &= (1-\eta)\, g_J(t,\bar a)\, J(t,\bar a). \end{aligned}$$

The overall dynamics is completed by the description of reproduction, which we obtain here through the usual age dependent fertility function $w = w(a)$ using the following nonlocal boundary condition

$$g_J(t,0)\, J(t,0) = \int_{\bar a}^{+\infty} w(\alpha)\, R(t,\alpha) \mathrm{d}\alpha.$$

In this connection, we recall the related results [10–12] in structured populations that take into consideration a juvenile–adult dynamics.

Once the biological evolution is defined, we introduce the income and cost functionals as follows. The income is related to the withdrawal of portions of the S population at given stages of its development. More precisely, we assume there are fixed ages $\bar a_1, \ldots, \bar a_N$, with $\bar a < \bar a_1 < \bar a_2 < \cdots < \bar a_N$, where the fractions $\vartheta_1, \ldots, \vartheta_N$ of the S population are kept, while the portions $(1-\vartheta_1), \ldots, (1-\vartheta_N)$ are sold. A very natural choice is to set $\vartheta_N \equiv 0$, meaning that nothing is left unsold after age $\bar a_N$. The dynamics of the whole system has then to be completed introducing the selection

$$S(t, \bar a_i+) = \vartheta_i\, S(t, \bar a_i-)$$

that takes place at the age $\bar a_i$, for $i = 1, \ldots, N$.

Summarizing, the dynamics of the structured (J, S, R) population is thus described by the following nonlocal system of balance laws, see also Fig. 1:

$$\begin{cases} \partial_t J + \partial_a \left(g_J(t,a)\, J\right) = d_J(t,a)\, J & (t,a) \in \mathbb{R}^+ \times [0,\bar a] \\ \partial_t S + \partial_a \left(g_S(t,a)\, S\right) = d_S(t,a)\, S & (t,a) \in \mathbb{R}^+ \times ([\bar a,+\infty[\setminus \{\bar a_1,\ldots,\bar a_N\}) \\ \partial_t R + \partial_a \left(g_R(t,a)\, R\right) = d_R(t,a)\, R & (t,a) \in \mathbb{R}^+ \times [\bar a,+\infty[\\ g_S(t,\bar a)\, S(t,\bar a) = \eta\, g_J(t,\bar a)\, J(t,\bar a) & t \in \mathbb{R}^+ \\ g_R(t,\bar a)\, R(t,\bar a) = (1-\eta)\, g_J(t,\bar a)\, J(t,\bar a) & t \in \mathbb{R}^+ \\ g_J(t,0)\, J(t,0) = \displaystyle\int_{\bar a}^{+\infty} w(\alpha)\, R(t,\alpha)\mathrm{d}\alpha & t \in \mathbb{R}^+ \\ S(t,\bar a_i+) = \vartheta_i\, S(t,\bar a_i-) & t \in \mathbb{R}^+, \quad i = 1,\ldots,N \\ J(0,a) = J_o(a) & a \in [0,\bar a] \\ S(0,a) = S_o(a) & a \in [\bar a,+\infty[\\ R(0,a) = R_o(a) & a \in [\bar a,+\infty[, \end{cases} \tag{1.1}$$

where we inserted the initial data (J_o, S_o, R_o).