Nonlinear Analysis: Real World Applications 36 (2017) 20–43

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Geometric approach to global asymptotic stability for the SEIRS models in epidemiology ☆

Guichen Lu [a,*], Zhengyi Lu [b]

[a] School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, China
[b] Department of Mathematics, Sichuan Normal University, Chengdu 610066, China

ARTICLE INFO

Article history:
Received 9 March 2016
Received in revised form 9 October 2016
Accepted 27 December 2016

Keywords:
SEIRS epidemic models
Invariant manifold
Global asymptotic stability
Li–Muldowney criterion

ABSTRACT

In this paper, we present a more general criterion for the global asymptotic stability of equilibria for nonlinear autonomous differential equations based on the geometric criterion developed by Li and Muldowney. By applying this criterion, we obtain some results for the global asymptotic stability of SEIRS models with constant recruitment and varying total population size. Based on these results, we give a complete affirmative answer to Liu–Hethcote–Levin conjecture. Furthermore, an affirmative answer to Li–Graef–Wang–Karsai's problem for SEIR model with permanent immunity and varying total population size is given.

1. Introduction

Epidemic models are used to study the transmission mechanisms of infectious diseases in host populations. They are important not only to reveal the diseases spreading mechanisms and to predict the future course of an outbreak, but also to evaluate optimal strategies to control an epidemic.

In 1906, Hamer developed a discrete time model to describe the recurrence of measles epidemics [1]. Ross, in 1911, introduced the first deterministic differential equation model to predict the dynamics of malaria transmission [2]. In 1927, Kermack and McKendrick [3] created the typical SIR epidemic model to study the spread of infectious disease. Furthermore, they proposed a threshold theory to determine whether an epidemic occurs or the disease simply dies out. More recently, studies of epidemic models have become one of the important areas in the mathematical theory of infectious diseases. There are many different types of epidemic

☆ Partially Supported by National Natural Science Foundation of China (Grant Nos. 11401062 and 11371386), Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20115134110001), Research Fund for the National Natural Science Foundation of Chongqing CSTC (Grant No. cstc2014jcyjA0080),Scientific and Technological Research Program of Chongqing Municipal Education Commission (Grant No. KJ1400937) and the Scientific Research Foundation of CQUT (Grant No. 2012ZD37).

* Corresponding author. Fax: +86 023 62563058.
E-mail address: guichenlu@cqut.edu.cn (G. Lu).

http://dx.doi.org/10.1016/j.nonrwa.2016.12.005

models for different problems, for example, in view of the routes of disease transmission, there exist models of SEI, SEIS, SIRS, SEIR, SEIR and SEIQR types. By considering factors like diversity on disease risk, age structure, seasonal effect, spatiotemporal effect, several models with time delay, age structure, seasonal effect and spatiotemporal effect have been introduced by researchers in the literature. A more detailed survey of these models is given by Capasso [4], Diekmann and Heesterbeek [5], Hethcote [6] and Martcheva [7].

The typical studies about disease dynamics in the literature have been focused on the global asymptotic stability issues. The most commonly used method for global asymptotic stability analysis in the previous studies is to use the Lyapunov stability theorem and LaSalle's invariance principle. This approach is successfully applied to the SIR, SEIR and SIRS models by Mena-Lorca and Hethcote [8], Korobeinikov [9], etc.

Busenberg and van den Driessche [10] developed the Bendixson–Dulac criterion in R^3. Together with Poincaré–Bendixson theory, the new criterion was applied by van den Driessche and Zeeman [11,12] to study the global asymptotic stability of three dimensional competitive Lotka–Volterra models. Based on additive compound matrix theory, Muldowney [13] and Li and Muldowney [14,15] generalized the Bendixson criterion in three and higher dimensional systems. Furthermore, they developed two methods for proving global asymptotic stability, that is, (1) proving global asymptotic stability using the Poincaré–Bendixson property, (2) proving global asymptotic stability using autonomous convergence theorems. The second method, sometimes quoted as Li–Muldowney geometric approach to global asymptotic stability, is successfully applied to determine the global asymptotic stability of epidemic models. For example, Li and his coworkers studied the global asymptotic stability of SEIR and SEIRS models [14,16–19]. Buonomo and his coauthors gave applications of the method to some epidemic models with bilinear incidence [20], as well as SIR and SEIR epidemic models with information dependent vaccination [21,22]. Zhang and Ma [23], Zhou and Cui [24] and Li et al. [25] made use of the approach in order to study the global dynamics of SEIR models. By modifying the Li–Muldowney geometric approach, Arino et al. [26] studied the global dynamics for an epidemic model with vaccination. Applications of Li–Muldowney geometric approach can also be found in population dynamics. Lu and Lu gave an application of the method to Lotka–Volterra models [27] and Gompterz models [28]. Ballyk et al. [29] applied their modified geometric approach to study a model of the chemostat and obtained new results.

The purpose of the present paper is to investigate the global asymptotic stability of SEIRS epidemic models. Liu et al. [30] showed that the dynamical behaviors of SEIRS models with nonlinear incidence $\beta S^q I^p$ can be different from those models with bilinear incidence βSI. They found in [30] that, if $p > 1$, there may exist multiple nontrivial endemic equilibria and periodic orbit; if $0 < p \leq 1$, they conjectured that the unique equilibrium may be globally asymptotically stable. This conjecture proposed by Liu et al. [30] has been studied by Li and Muldowney [14], Li, Muldowney and van den Driessche [17] and Cheng and Yang [31].

Li et al. [19] investigated a SEIR model that incorporates varying total population and disease caused death. Global asymptotic stability of the endemic equilibrium was proved under the restriction of the disease-caused death rate and mean latent period in [19]. Furthermore, by virtue of numerical simulation, they suggested that the restriction of disease caused rate was unnecessary. Whether or not the restriction is necessary for the global asymptotic stability is unresolved in [19].

In the present paper, we will propose a more general criterion for showing the global asymptotic stability of nonlinear autonomous systems. This generalizes the criteria developed by Li and Muldowney [32,33,14] and will allow us to resolve the Liu–Hethcote–Levin conjecture completely and remove the restriction of disease caused rate in [19].

An outline of this paper is as follows. We establish a geometric criterion in Section 2, the global asymptotic stability of SEIRS model with constant recruitment and Liu–Hethcote–Levin conjecture are shown in Section 3, the result for global asymptotic stability of SEIR model with varying total population and an