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Positive solutions for the fractional Laplacian in the almost critical case in a bounded domain

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ABSTRACT

We prove existence of multiple positive solutions for a *fractional scalar field equation* in a bounded domain, whenever p tends to the critical Sobolev exponent. By means of the "photography method", we prove that the topology of the domain furnishes a lower bound on the number of positive solutions.

1. Introduction

In the celebrated papers [1,2] Benci, Cerami and Passaseo proved an existence result of positive solutions of the following problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, $N \geq 3$ and $p < 2^* = \frac{2N}{N-2}$, the critical Sobolev exponent of the embedding of $H_0^1(\Omega)$ in the Lebesgue spaces. Roughly speaking they show that (among other results), for p near 2^*, the number of positive solutions is bounded from below by a topological invariant associated to Ω. More specifically they prove the following.

Theorem. *There exists a* $\bar{p} \in (2, 2^*)$ *such that for every* $p \in [\bar{p}, 2^*)$ *problem* (1.1) *has (at least)* $\operatorname{cat}_{\bar{\Omega}}(\bar{\Omega})$ *positive solutions. Even more, if* Ω *is not contractible in itself, the number of solutions is* $\operatorname{cat}_{\bar{\Omega}}(\bar{\Omega}) + 1$.

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Hereafter given a topological pair $A \subset X, \mathrm{cat}_X(A)$ is the Lusternik–Schnirelmann category of the set A in X (see e.g. [3]).

To prove this result, the authors used variational methods: an energy functional related to the problem is introduced in such a way that the solutions are seen as critical point of this functional restricted to L^p-ball. Then the "photography method" (which permits to see a photography of the domain Ω in a suitable sublevel of the functional) is implemented in order to prove the existence of many critical points by means of the classical Lusternik–Schnirelmann Theory.

The aim of this paper is to prove the fractional counterpart of the theorem above. Indeed, due to the large literature appearing in these last years on fractional operators, it is very natural to ask if a similar result also holds for the fractional Laplacian. In other words we consider in this paper the following nonlocal problem

$$\begin{cases} (-\Delta)^s u + u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where $s \in (0,1)$, $p \in (2, 2_s^*)$ with $2_s^* := 2N/(N-2s), N > 2s$.

The operator $(-\Delta)^s$ is the *fractional Laplacian* which is defined by

$$(-\Delta)^s u(x) := C(N,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N$$

for a suitable constant $C(N,s) > 0$ whose exact value is not really important for our purpose. This normalization constant enters whenever one want to recover the usual Laplacian when $s \to 1$, for details we refer the reader to [4, Section 4]. The Dirichlet condition in (1.2) is then given on $\mathbb{R}^N \setminus \Omega$ reflecting the fact that $(-\Delta)^s$ is a nonlocal operator.

Before to state our result, let us introduce few basic notations. For a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ let

$$[u]^2_{D^{s,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy$$

be the (squared) *Gagliardo seminorm* of u. Let us define the Hilbert space

$$D^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2_s^*}(\mathbb{R}^N) : [u]^2_{D^{s,2}(\mathbb{R}^N)} < +\infty \right\},$$

which is continuously embedded into $L^{2_s^*}(\mathbb{R}^N)$. Let finally

$$D_0^{s,2}(\Omega) = \left\{ u \in D^{s,2}(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

From now on it will be convenient to adopt the following convention: functions defined in a subset of $\mathbb{R}^N$, let us say A, will be thought extended by zero on $\mathbb{R}^N \setminus A$, whenever regarded as functions defined on the whole $\mathbb{R}^N$.

Note that being $\partial\Omega$ smooth, $D_0^{s,2}(\Omega)$ can be also defined as the completion of $C_0^\infty(\Omega)$ under the norm $[\cdot]_{D^{s,2}(\mathbb{R}^N)}$. Moreover, it is $D_0^{s,2}(\Omega) = \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega\}$.

Recall that we have the continuous embedding $D_0^{s,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \le p \le 2_s^*$ and that the embedding is compact for $1 \le p < 2_s^*$.

We then say that $u \in D_0^{s,2}(\Omega)$ is a solution (in the distributional sense) of (1.2) if

$$\forall\, v \in D_0^{s,2}(\Omega) : \quad \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} uv dx = \int_{\mathbb{R}^N} |u|^{p-2} uv dx. \tag{1.3}$$

The main result of the paper gives a positive answer on the possibility of extending the Benci, Cerami and Passaseo result to the fractional case.