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Melnikov functions and limit cycles in piecewise smooth perturbations of a linear center using regularization method

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This article is dedicated to Professor José Augusto Baêta Segundo on his 65th birthday

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A B S T R A C T

In this article we study limit cycles in piecewise smooth perturbations of a linear center. In this setting it is common to adapt classical results for smooth systems, like Melnikov functions, to non-smooth ones. However, there is little justification for this procedure in the literature. By using the regularization method we give a theoretical proof that supports the use of Melnikov functions directly from the original non-smooth problem.

1. Introduction and statement of the results

The main subjects studied in this article are the existence and positions of limit cycles in piecewise smooth perturbations of a linear center. Consider

$$X' = \frac{dX}{dt} = Z(X,\varepsilon) = \begin{cases} X^-(X) = AX + \varepsilon G^-(X), & y \le 0, \\ X^+(X) = AX + \varepsilon G^+(X), & y \ge 0, \end{cases} \tag{1}$$

where $X = (x, y) \in \mathbb{R}^2, \varepsilon \ge 0$ is a small parameter, the prime denotes derivative with respect to the independent variable t, called here the time,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2}$$

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$$
\begin{aligned}
G:\mathbb{R}^2 &\longrightarrow \mathbb{R}^2\\
X &\longmapsto G(X)=\begin{cases} G^-(X)=(g_1^-(x,y),g_2^-(x,y)), \ y\le 0,\\ G^+(X)=(g_1^+(x,y),g_2^+(x,y)), \ y\ge 0,\end{cases}
\end{aligned} \tag{3}
$$

with $G^\pm$ smooth vector fields (class $C^k, k\ge 1$) and $G^\pm(0,0)=(0,0)$.

There are many papers studying the bifurcation problem of limit cycles from the period annulus of system (1) when $\varepsilon=0$. Two main tools are underlying to these studies: Melnikov Functions and Averaging Theory. For the first case see [1–3] and for the second one see [4,5] and references therein.

To the best of our knowledge in all the articles concerning the above problem Melnikov functions or averaging theory were used directly from the non-smooth problem (1). In this article we propose to use the regularization method in order to study limit cycles in system (1) when $\varepsilon>0$. In fact, we study the equivalence of both methods.

First of all, we study limit cycles of the discontinuous system (1) using a type of Melnikov function given in the following theorem.

Theorem 1. *If $a>0$ is a simple zero of the function*

$$
\mathcal{M}(a)=\mathcal{M}^+(a)+\mathcal{M}^-(a), \tag{4}
$$

where

$$
\mathcal{M}^+(a)=\int_0^\pi \left[g_1^+(a\cos(s),a\sin(s))\cos(s)+g_2^+(a\cos(s),a\sin(s))\sin(s)\right]\mathrm{d}s
$$

and

$$
\mathcal{M}^-(a)=\int_\pi^{2\pi} \left[g_1^-(a\cos(s),a\sin(s))\cos(s)+g_2^-(a\cos(s),a\sin(s))\sin(s)\right]\mathrm{d}s,
$$

then for $\varepsilon>0$ sufficiently small there exists a limit cycle X^ε of (1) such that X^ε tends to the circle with center at the origin and radius a when ε goes to 0. The limit cycle is stable if $\mathcal{M}'(a)<0$ and unstable if $\mathcal{M}'(a)>0$.

Theorem 1 is proved in Section 2 and its proof is based on the study of the Taylor development of the displacement function defined by the Poincaré first return map and the direct use of Implicit Function Theorem and differentiability of solutions of ordinary differential equations with respect to parameters and initial conditions.

Another approach in order to study limit cycles of the discontinuous system (1) is based on the regularization method, which we describe briefly now. See Section 3 for details.

Consider $\mathcal{H}:\mathbb{R}^2\to\mathbb{R}, \mathcal{H}(x,y)=y$. It is easy to see that zero is a regular value of the smooth function $\mathcal{H}$. Define the sets

$$
\Sigma=\mathcal{H}^{-1}(0),\qquad \Sigma^-=\mathcal{H}^{-1}(-\infty,0),\qquad \Sigma^+=\mathcal{H}^{-1}(0,+\infty).
$$

Then $\mathbb{R}^2=\Sigma^-\cup\Sigma\cup\Sigma^+$. The set Σ is called the separation line between the two zones Σ^- and Σ^+.

From a C^k function $\varphi:\mathbb{R}\to\mathbb{R}$, called a transition function, defined by $\varphi(t)=0$, if $t\le -1, \varphi(t)=1$, if $t\ge 1$ and $\varphi'(t)>0$, if $t\in(-1,1)$, and a real number $\mu>0$, called regularization parameter, we define the function φ^μ by $\varphi^\mu(t)=\varphi(t/\mu)$, for all $t\in\mathbb{R}$.

A regularization of Z in (1) produces a two parameter family of smooth vector fields

$$
Z^\mu(X,\varepsilon)=(1-\varphi^\mu(y))X^-(x,y)+\varphi^\mu(y)X^+(x,y)=AX+\varepsilon R(X,\mu), \tag{5}
$$

where

$$
R(X,\mu)=(r_1(x,y,\mu),r_2(x,y,\mu))
$$