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Weak solutions of fractional differential equations in non cylindrical domains

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HIGHLIGHTS

- A time fractional diffusion equation in non cylindrical domains is studied.
- The Galerkin method is used to obtain a weak solution.
- Special approximate solutions are defined to obtain a priori estimates.
- A notion of a weak solution is worked out.

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ABSTRACT

We study a time fractional heat equation in a non cylindrical domain. The problem is one-dimensional. We prove existence of properly defined weak solutions by means of the Galerkin approximation.

1. Introduction

1.1. Motivation

In this paper we study the heat equation with the Caputo time derivative in a non cylindrical domain. It is intended to be the first in a series devoted the Stefan problem with fractional derivatives. We address here

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a simple, yet non-trivial question of the existence of solutions to the problem where the interfacial curve is given to us.

The interest in fractional PDE's stems from many sources. One of them is the theory of stochastic processes admitting jumps and continuous paths, see [1–3]. Our motivation comes from phenomenological models of sediment transport, see [4–8]. Apparently, this problem awaits a systematic treatment.

In these problems, the position of the advancing front s is not known, i.e., it is a part of the problem. Here, we consider a simplified situation,

$$\begin{cases} D_s^\alpha u(x,t) &= u_{xx}(x,t) + f(x,t) & \text{for} \quad 0 < x < s(t), 0 < t < T, \\ -u_x(0,t) &= h(t) & \text{for} \quad 0 < t < T, \\ u(s(t),t) &= 0 & \text{for} \quad 0 < t < T, \\ u(x,0) &= u_0(x) & \text{for} \quad 0 < x < b = s(0), \end{cases} \tag{1.1}$$

where $h(t)$, $f(x,t)$, $u_0(x)$, s are given functions and s is nondecreasing. By D_s^α we understand the Caputo fractional derivative, defined as follows

$$D_s^\alpha w(x,t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-\tau)^{-\alpha} w_t(x,\tau) d\tau & \text{for} \quad x \le s(0), \\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_{s^{-1}(x)}^t (t-\tau)^{-\alpha} w_t(x,\tau) d\tau & \text{for} \quad x > s(0). \end{cases} \tag{1.2}$$

In the original free boundary problem the above system is augmented by an equation governing the evolution of s, here s is given to us. We assume that s is increasing. We will also use the following convention: if $x > b$, then $s^{-1}(x) = \max\{t : s(t) = x\}$ and if $x \in [0,b]$, then $s^{-1}(x) = 0$.

Experience with fractional derivatives tells us that tools used for PDE's require substantial modification before they can be applied to equations with fractional derivatives. In case of time fractional derivatives this means that we have to take into account the whole history of the process. This is the main difficulty of the analysis. This is why we set limited goals in this paper.

Our main result, expressed in Theorem 1.1, is the existence of properly defined weak solutions to (1.1). We use for this purpose the Galerkin method. This is a straightforward approach in case of PDE's, however, here it gets complicated when applied to (1.1). Now, we briefly describe the content of our paper. In Section 2 we construct approximate solutions by means of the Galerkin method. In Section 3 we derive the necessary estimate and show that we can pass to the limit. This process yields an existence of a weak solution. Proofs of technical results are presented in the Appendix.

1.2. Preliminaries and a definition of a weak solution

Here, we will make our assumptions, upheld throughout this paper. We will denote by $\alpha \in (0,1)$ the order of the Caputo derivative. By s we denote the position of the interface. Its initial position $b = s(0)$ is positive. Function s is not only increasing and continuous, but also

$$t \mapsto t^{1-\alpha} \dot{s}(t) \in C([0,T]) \quad \text{and} \quad \dot{s} \ge 0. \tag{1.3}$$

With the help of s, we define a non cylindrical domain $Q_{s,t}$ by the following formula,

$$Q_{s,t} = \{(x,\tau) : 0 < x < s(\tau), 0 < \tau < t\}.$$

The fractional derivative D_s^α is a complicated operator. In order to simplify the analysis, we introduce an auxiliary integral operator for functions defined on the domain $Q_{s,t}$ by the following formula,

$$I_s^\alpha w(x,t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} w(x,\tau) d\tau & \text{for} \quad x \le s(0), \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s^{-1}(x)}^t (t-\tau)^{\alpha-1} w(x,\tau) d\tau & \text{for} \quad x > s(0). \end{cases} \tag{1.4}$$