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Existence of solutions to the nonlinear, singular second order Bohr boundary value problems

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ABSTRACT

This paper investigates the existence of solutions for nonlinear systems of second order, singular boundary value problems (BVPs) with Bohr boundary conditions. A key application that arises from this theory is the famous Thomas–Fermi equations for the model of the atom when it is in a neutral state. The methodology in this paper uses an alternative and equivalent BVP, which is in the class of resonant singular BVPs, and thus this paper obtains novel results by implementing an innovative differential inequality, Lyapunov functions and topological techniques. This approach furnishes new results in the area of singular BVPs for *a priori* bounds and existence of solutions, where the BVP has unrestricted growth conditions and subject to the Bohr boundary conditions. In addition, the results can be relaxed and hold for the non-singular case too.

1. Introduction

This paper investigates the existence of solutions and *a priori bounds* to systems of two-point nonlinear second order singular differential equations. The main problem discussed is the following differential equation with Bohr boundary conditions:

$$\frac{1}{p(t)}(p(t)\mathbf{y}'(t))' = q(t)\mathbf{f}(t, \mathbf{y}(t), p(t)\mathbf{y}'(t)), \quad t \in (0, T) \tag{1.1}$$

$$\mathbf{y}(0) = \mathbf{c}; \qquad \int_0^T \frac{ds}{p(s)} \lim_{t\to T^-} p(t)\mathbf{y}'(t) - \mathbf{y}(T) = \mathbf{d}. \tag{1.2}$$

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In the above, the function $\mathbf{f} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ are vector-valued constants and

$$p \in C([0,T];\mathbb{R}) \cap C^1((0,T);\mathbb{R}) \quad \text{with } p > 0 \text{ on } (0,T) \tag{1.3}$$

$$q \in C((0,T);\mathbb{R}) \quad \text{with } q > 0 \text{ on } (0,T). \tag{1.4}$$

The motivation behind the study of this problem comes from the significant applications to real world modelling, mainly, the Thomas–Fermi equation [1,2] for determining the electrical potential of a neutral atom with radius T given by

$$y''(t) = t^{-1/2} y^{3/2}, \quad t \in (0,T) \tag{1.5}$$

$$y(0) = 1; \qquad Ty'(T) - y(T) = 0. \tag{1.6}$$

This boundary value problem (BVP) is singular and nonlinear in nature since it contains a singularity in time at $t = 0$ and is nonlinear in y. In addition, at the right-end point, the boundary condition is of the Bohr type. This boundary condition adds a level of complexity, especially in problems where $\mathbf{f}$ is not totally bounded. This causes the BVP to be considered as an alternative problem, a singular BVP at resonance, since the equivalent integral representation operator for original BVP is non-invertible. The technique that is used herein has been 'coined' as a shift argument, and a similar variation has been used in Infante, Pietramala & Tojo [3], Webb & Zima [4] and Al Mosa & Eloe [5]. The understanding of a solution to BVP (1.1), (1.2) is a function $\mathbf{y} \in C^2((0,T);\mathbb{R}^n) \cap C([0,T];\mathbb{R}^n)$, $p\mathbf{y}' \in C([0,T];\mathbb{R}^n)$ that satisfies (1.1), (1.2). Also, the norm on a vector $\mathbf{u} \in \mathbb{R}^n$ is defined by $\|\mathbf{u}\| := \langle \mathbf{u}, \mathbf{u} \rangle^{1/2}$ where $\langle \cdot, \cdot \rangle$ is the usual Euclidean dot product in $\mathbb{R}^n$.

The development of the theory of singular BVPs grew rapidly from the mid 1980's with authors such as Agarwal & O'Regan [6–9], leading the field with numerous papers and books on the field. Although, this particular type of BVP has hardly been discussed in the literature and more attention has been put on the problem when the boundary conditions are Dirichlet, Mixed or Sturm–Liouville, see [6,10,9] and the references therein. In addition, the method used to begin tackling this problem is to consider the problem as a BVP at resonance. The theory of resonant problems has gauged a lot of recent notable interest in both the singular case, O'Regan [11,12] and in the non-singular case with different boundary conditions in [5,13,14,3,15,16,4].

The problem has been discussed previously (analytically) in the literature by Agarwal, O'Regan & Palamides [17,18] but in both papers they use an upper and lower solution method and only on the scalar case ($n = 1$) to determine the existence of solutions to the BVP (1.5), (1.2). This particular Thomas–Fermi BVP has been investigated numerically as well with contributions from [19–21]. The results in our paper address the general situation for systems of equations, provides bounds on the solutions and their derivatives, as well as provides a new alternative method for determining the existence of solutions via a singular differential inequality. This inequality is inspired by the work of Hartman [22] and George & York [23]. One of the motivating reasons for research in this area which appears in this paper is that we prove novel results for the generalised form of the BVP (1.1), (1.2) and we conclude by applying the theory to the famous Thomas–Fermi BVP.

The first section investigates the general problem and presents three main results, the construction of the equivalent integral representation and two results regarding the existence of solutions to (1.1), (1.2). The final section of the paper is devoted to solving the famous Thomas Fermi problem with Bohr boundary conditions, (1.5), (1.6). This is the main motivation of this paper and to provide the theory to solve the general problem, (1.1), (1.2).

In the work of O'Regan [8], the BVP (1.1), (1.2) is discussed in the scalar case ($n = 1$) and he made the assumption that all solutions and their derivatives are bounded. In addition, he made the assumption that