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Nonexistence and multiplicity of solutions for nonlinear elliptic systems in $\mathbb{R}^N$ ☆

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ABSTRACT

In the whole space $\mathbb{R}^N$, we will study the following nonlinear elliptic system in two cases:

$$\begin{cases} -\triangle u + V_1(x)u = f(x,u,v), & x \in \mathbb{R}^N, \\ -\triangle v + V_2(x)v = g(x,u,v), & x \in \mathbb{R}^N, \\ u(x) \to 0, \ v(x) \to 0, & |x| \to \infty. \end{cases}$$

Case 1: The periodic case (i.e., V_1, V_2, f and g are periodic in x_i for $i = 1, \dots, N$), we obtain the nonexistence of nontrivial solutions for the system. The existence result has been obtained in Chen and Ma (2013). **Case 2**: The non-periodic case (i.e., V_1, V_2, f and g are non-periodic), we mainly focus on the case where the nonlinearities f and g are superlinear at infinity, and we obtain infinitely many nontrivial solutions of the system by variational methods. To the best of our knowledge, there is **no work** focusing on the system in Cases 1–2. **The main novelties** of this paper can be summarized as follows: (1) The system is defined in the whole space $\mathbb{R}^N$; (2) The potentials $V_i(x)$ $(i = 1, 2)$ can be sign-changing; (3) The nonexistence of nontrivial solutions in the periodic case is obtained; (4) Infinitely many nontrivial solutions in the non-periodic case are obtained.

1. Introduction and main results

In the whole space $\mathbb{R}^N$, we consider the following seconder order nonlinear elliptic system

$$\begin{cases} -\triangle u + V_1(x)u = f(x,u,v), & x \in \mathbb{R}^N, \\ -\triangle v + V_2(x)v = g(x,u,v), & x \in \mathbb{R}^N, \\ u(x) \to 0, \ v(x) \to 0, & |x| \to \infty, \end{cases} \tag{1.1}$$

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where the potentials $V_1, V_2 \in C(\mathbb{R}^N, \mathbb{R})$ and the nonlinearities $f, g \in C(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$. The nonlinearities (f, g) are the gradient of some function, i.e., there is a function $F(x, U) \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ such that $\nabla F = (f, g)$, where ∇F denotes the gradient of F in $U = (u, v) \in \mathbb{R}^2$.

If $V_1(x) \equiv -\xi_0$ and $V_2(x) \equiv -\zeta_0$ $(\xi_0, \zeta_0 \in \mathbb{R})$, then (1.1) reduces to

$$\begin{cases} -\triangle u = \xi_0 u + f(x, u, v), & x \in \Omega, \\ -\triangle v = \zeta_0 v + g(x, u, v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^{\mathrm{N}}$. The system (1.2) has been studied by many authors [1–10] in the cases where nonlinearities are asymptotically linear, superlinear and sublinear at infinity. The authors [1,6,7,9] established some existence results, and the authors [2–6,8–10] obtained infinitely many solutions under the oddness and some assumptions on nonlinearities. Recently, Chen [4] improved the superlinear results and obtained infinitely many solutions.

Remark 1.1. For (1.1) in the whole space $\mathbb{R}^N$, we are interested in the following two cases:
Case 1 (The periodic case, i.e., V_1, V_2, f and g are periodic in x_i for $i = 1, \ldots, N$). In the periodic case, we should mention that the only published work on the system (1.1) is the paper [11]. The authors [11] obtained the existence of nontrivial solutions of (1.1) by using a variant generalized weak linking theorem. Therefore, we shall study the nonexistence of nontrivial solutions for (1.1).
Case 2 (The non-periodic case, i.e., V_1, V_2, f and g are non-periodic). We mainly focus on the case where the nonlinearity $F(x, U)$ is superquadratic as $|U| \to \infty$, and we study the existence of infinitely many nontrivial solutions for (1.1) by the variational method.

To the best of our knowledge, there is no work focusing on the system (1.1) in both Case 1 and Case 2.

Part 1. *The periodic case.*

Let $L_i := -\triangle + V_i$, $i = 1, 2$. We are interested in the strongly indefinite case, i.e.,
$(\mathbf{L_1})$ $V_i(x)$ are 1-periodic in x_j $(j = 1, \ldots, N)$ and

$$\underline{\mu}_i := \sup(\sigma(L_i) \cap (-\infty, 0)) < 0 < \overline{\mu}_i := \inf(\sigma(L_i) \cap (0, \infty)), \quad i = 1, 2.$$

Let $|\cdot|$ and $(\cdot, \cdot)$ denote the usual norm and inner product in $\mathbb{R}^2$, respectively. Let

$$\rho_0 := \min\{\min\{-\underline{\mu}_1, \overline{\mu}_1\}, \min\{-\underline{\mu}_2, \overline{\mu}_2\}\} = \min\{-\underline{\mu}_1, \overline{\mu}_1, -\underline{\mu}_2, \overline{\mu}_2\},$$

and

$$H(x, U) := \frac{1}{2}(\nabla F(x, U), U) - F(x, U), \ \forall (x, U) \in \mathbb{R}^N \times \mathbb{R}^2.$$

Now, our nonexistence result reads as follows:

Theorem 1.1. *Assume that* (L_1) *and the following conditions hold*
$(\mathbf{F_1})$ $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ *is* 1*-periodic in* x_j $(j = 1, \ldots, N)$*, and*

$$|\nabla F(x, U)| = o(|U|) \quad \text{as} \quad |U| \to 0 \quad \text{uniformly in } x.$$

$(\mathbf{F_2})$ $F(x, U) \geq 0$*, and* $\frac{1}{2}(\nabla F(x, U), U) > F(x, U)$ *if* $U \in \mathbb{R}^2 \backslash \{0\}$*,* $\forall (x, U) \in \mathbb{R} \times \mathbb{R}^2$*.*
$(\mathbf{F_3})$ $F(x, U) = \frac{1}{2}\eta|U|^2 + G(x, U)$*, where* $\eta \in (0, +\infty)$*,* $G \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ *and*

$$\frac{|\nabla G(x, U)|}{|U|} \to 0 \quad \text{as } |U| \to \infty \text{ uniformly in } x.$$