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Algebraic traveling waves for the generalized Newell–Whitehead–Segel equation

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ABSTRACT

In this paper we provide the only possible algebraic traveling wave solutions for the celebrated general Newell–Whitehead–Segel equation.

1. Introduction and statement of the main results

The problems of the propagation of nonlinear waves have fascinated scientists for over two hundred years. The modern theory of nonlinear waves, like many areas of mathematics, had its beginnings in attempts to solve specific problems, the hardest among them being the propagation of waves in water. There was significant activity on this problem in the 19th century and the beginning of the 20th century, including the classic work of Stokes, Lord Rayleigh, Korteweg and de Vries, Boussinesque, Benard and Fisher to name some of the better remembered examples [1,2]. One particularly noteworthy contribution was the explosion of activity unleashed by the numerical discovery of the soliton by Zabusky and Kruskal in the early sixties, and the earliest theoretical explanation by Gardner, Greene, Kruskal, and Miura in the latter part of that decade, which subsequently led to the present-day theory of integrable partial differential equations. Nonlinear waves and coherent structures is an interdisciplinary area that has many important applications, including nonlinear optics, hydrodynamics, plasmas and solid-state physics. In fact, for any physical system where the dynamics is driven by, and mainly determined by, phase coherence of the individual waves, it has applications and consequences. Modern theories describe nonlinear waves and coherent structures in a diverse variety of fields, including general relativity, high energy particle physics, plasmas, atmosphere

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and oceans, animal dispersal, random media, chemical reactions, biology, nonlinear electrical circuits, and nonlinear optics. For example, in the latter, the mathematics developed for describing the propagation of information via optical solitons is most striking, attaining an incredible accuracy. It has been experimentally verified and spans twelve orders of magnitude: from the wavelength of light to transoceanic distances. It also guides the practical applications in modern telecommunications. Many other nonlinear wave theories mentioned above can claim similar success.

Nowadays it has been universally acknowledged in the physical, chemical and biological communities that the reaction–diffusion equation plays an important role in dissipative dynamical systems. Typical examples are provided by the fact that there are many phenomena in biology where a key element or precursor of a developmental process seems to be the appearance of a traveling wave of chemical concentration (or mechanical deformation). When reaction kinetics and diffusion are coupled, traveling waves of chemical concentration can effect a biochemical change much faster than straight diffusional processes. This usually gives rise to reaction–diffusion equations.

The simplest reaction–diffusion equation is in one spatial dimension in plane geometry,

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + R(u),$$

is also referred to as the KPP (Kolmogorov–Petrovsky–Piskounov) equation [3]. If the reaction term vanishes, then the equation represents a pure diffusion process. The corresponding equation is Fick's second law. The choice $R(u) = u(1-u)$ yields Fisher's equation that was originally used to describe the spreading of biological populations [4]. The Newell–Whitehead–Segel equation with the choice $R(u) = u(1-u^2)$ is to describe Rayleigh–Benard convection [5], the more general Zeldovich equation with $R(u) = u(1-u)(u-a)$ and $0 < \alpha < 1$ that arises in combustion theory [6].

Most of these nonlinear differential equations do not have an analytical solution. The analytical solutions of such equations are of fundamental importance. Among the possible solutions, the traveling wave solutions that are solutions which do not change their shape, and that propagate at constant speed have been widely studied. In this paper we focus in the more general form of the Newell–Whitehead–Segel equation.

The Newell–Whitehead–Segel equation models the interaction of the effect of the diffusion term with the nonlinear effect of the reaction term. The Newell–Whitehead–Segel equation is written as:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu - du^{n+1},$$

where $a > 0, b, d$ are real parameters and n is an integer. The first term on the left hand side, $\partial u/\partial t$, expresses the variations with time at a fixed location, the first term on the right hand side, $\partial^2 u/\partial x^2$, expresses the variations with spatial variable x at a specific time and the remaining terms on the right hand side, $bu - du^{n+1}$, takes into account the effect of the source term. It can be thought of as the (nonlinear) distribution of temperature in an infinitely thin and long rod or as the flow velocity of a fluid in an infinitely long pipe with small diameter. The Newell–Whitehead–Segel equations have wide applicability in mechanical and chemical engineering, ecology, biology and bio-engineering.

There are various approaches for constructing traveling wave solutions. Some of these approaches are the Jacobi elliptic function method [7], inverse scattering method [8], Hirotas bilinear method [9], homogeneous balance method [10], homotopy perturbation method [11], Weierstrass function method [12], symmetry method [13], Adomian decomposition method [14], sine/cosine method [15], tan/coth method [16], the Exp-function method [17], etc. But most of the methods may sometimes fail or can only lead to a kind of special solution and the solution procedures become very complex as the degree of the nonlinearity increases. In [18] the authors gave a technique to prove the existence of traveling wave solutions for general nth order partial differential equations by showing that traveling wave solutions exist if and only if the associated n-dimensional first order ordinary differential equation has some invariant algebraic curve. In this paper we will consider only the case $n = 2$.