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Determining form and data assimilation algorithm for weakly damped and driven Korteweg–de Vries equation — Fourier modes case

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ABSTRACT

We show that the global attractor of a weakly damped and driven Korteweg–de Vries equation (KdV) is embedded in the long-time dynamics of an ordinary differential equation called a determining form. In particular, there is a one-to-one identification of the trajectories in the global attractor of the damped and driven KdV and the steady state solutions of the determining form. Moreover, we analyze a data assimilation algorithm (down-scaling) for the weakly damped and driven KdV. We show that given a certain number of low Fourier modes of a reference solution of the KdV equation, the algorithm recovers the full reference solution at an exponential rate in time.

1. Introduction

The Korteweg–de Vries equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1.1}$$

was derived as a model of unidirectional propagation of water waves with small amplitude in a channel. It was first introduced by Boussinesq and then reformulated by Diederik Korteweg and Gustav de Vries in 1885. The function $u(x,t)$ in (1.1) represents the elongation of the wave at time t and position x. The solutions of this nonlinear and dispersive equation are solitary waves. In physical applications, however, one

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can expect some dissipation of energy, as well as external excitation. To account for these effects, damping and forcing terms are added to the model

$$u_t + uu_x + u_{xxx} + \gamma u = f. \tag{1.2}$$

Existence and the uniqueness of the solution of the damped and driven Korteweg–de Vries (KdV) equation subject to the boundary conditions

$$u(t, x) = u(t, x + L), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.3}$$

can be shown by adjusting the methods used for undamped KdV in [1,2] or [3]. The existence of the weak global attractor $\mathcal{A}$, for (1.2)–(1.3), was shown in [4], and the strong global attractor in H^2 was shown in [5]. In particular, it has been shown in [5] that there exists a constant R, that depends only on γ and $|f|_{H^2}$, such that

$$\sup_{s\in\mathbb{R}} |u(s)|_{H^2} \le R, \tag{1.4}$$

for every $u(\cdot) \subset \mathcal{A}$. We observe that the estimates detailed in Section 4, can be followed almost line by line in order to obtain an explicit bound for R.

For many strongly dissipative PDE's, capturing the attractor by a finite system of ordinary differential equations is achieved by restricting the equation to an inertial manifold, as is done for Kuramoto–Sivashinsky, Ginzburg–Landau and certain reaction–diffusion equations (see, e.g., [6] and references therein). An inertial manifold is a finite dimensional Lipschitz positively invariant manifold which attracts all the solutions at an exponential rate. A sufficient condition for the existence of an inertial manifolds is the presence of large enough gaps in the spectrum of the linear dissipative operator, i.e. the presence of separation of scales in the underlying dynamics. The existence of inertial manifolds is still out of reach for various dissipative equations, including the two-dimensional Navier–Stokes equations, and the damped and driven KdV equations (1.2)–(1.3). Our aim here is to capture the attractor in H^2, of the damped and driven KdV, by the dynamics of an ordinary differential equation, called a *determining form*, which is defined in the phase space of trajectories.

A determining form is found for the 2D Navier–Stokes equations (NSE) in [7] by using finitely many determining modes. In that work, the trajectories in the attractor of the 2D NSE are identified with traveling wave solutions of the determining form. Another type of determining form is found for the 2D NSE by the same authors in [8]. The steady state solutions of this second kind of determining form are precisely the trajectories in the global attractor of the 2D NSE. Dissipativity (viscosity) plays a fundamental role in establishing a determining form for this equation.

In contrast, the weakly damped and driven nonlinear Schrödinger equation (NLS) and weakly damped and driven KdV are dispersive equations. They are not strongly dissipative due to the absence of viscosity. To embed the attractors of these systems in the long time dynamics of ordinary differential equations requires different techniques. Recently, we have shown that a determining form of the second kind exists for the damped and driven NLS (see [9]) using a feedback control term involving the Fourier projection of a trajectory in the attractor. In this paper we adapt the approach in [8,9] for the KdV. As in [9] the analysis here uses compound functionals motivated by the Hamiltonian structure of the corresponding systems.

The idea for determining forms starts with the property of determining modes (see [10]). A projector P is said to be determining if whenever $u_1(\cdot), u_2(\cdot) \subset \mathcal{A}$ have the same projection $Pu_1(t) = Pu_2(t)$ for all $t \in \mathbb{R}$, they are in fact the same solution. A determining projector P defines a map W on the set $\mathcal{S} = \{Pu(\cdot)|u(\cdot) \subset \mathcal{A}\}$. A key step in constructing a determining form is to extend this map to a function space. If $P = P_N$ is the projection onto the first N Fourier modes, the number N is called the number of determining modes. Like the dimension of $\mathcal{A}, N$ serves as a measure of the complexity of the flow, and the