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Harmonic functions with nonlinear Neumann boundary condition and their Morse indices

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Dedicated to the memory of Professor Abbas Bahri

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ABSTRACT

We consider the solutions of a nonlinear Neumann elliptic equation $\Delta u = 0$ in Ω, $\partial u/\partial \nu = f(x,u)$ on $\partial\Omega$, where Ω is a bounded open smooth domain in $\mathbb{R}^N$, $N \geq 2$ and f satisfies super-linear and subcritical growth conditions. We prove that L^∞-bounds on solutions are equivalent to bounds on their Morse indices.

1. Introduction and main results

The purpose of this paper is to study the nonlinear Neumann elliptic problem

$$(P)\ \begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = f(x,u) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open smooth domain in $\mathbb{R}^N$, $\partial/\partial\nu$ denotes the derivative with respect to the outward normal to $\partial\Omega (N \geq 2), f(x,t)$ is continuous on $\partial\Omega \times \mathbb{R}$, differentiable with respect to t; and $\partial f/\partial t$ is continuous on $\partial\Omega \times \mathbb{R}$.

Elliptic problem with nonlinear boundary condition like (P) has been widely studied in the past by many authors and it is still an area of intensive research. This kind of boundary condition appears in a rather

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natural way in some physical models. For example, problem (P) can be thought of as a model for heat propagation. In this case u stands for the stationary temperature and the normal derivative $\frac{\partial u}{\partial \nu}$ that appears in the boundary condition represents the heat flux. Hence the boundary condition represents a nonlinear radiation law at the boundary.

Here, we will deal with the super-linear case, that is,

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{t} = +\infty, \quad \text{uniformly on } x \in \partial\Omega, \tag{1.1}$$

with a "subcritical" growth

$$|f(x,t)| \le a(1+|t|^p), \quad a>0, \text{ for all } (x,t) \in \partial\Omega \times \mathbb{R}, \tag{1.2}$$

where p satisfies

$$1 < p < N/(N-2) \quad \text{if } N \ge 3 \text{ and } p \in (1,\infty) \quad \text{if } N = 2. \tag{1.3}$$

We remark that the assumptions (1.1)–(1.3) imply that we are in a nonlinear and subcritical case. Hence, using the compactness of the embedding of $H^1(\Omega)$ into $L^{p+1}(\partial\Omega)$, the variational methods imply that the problem (P) has solutions.

In this paper, we assume that f satisfies

$$\lim_{|t|\to+\infty} f(x,t)t^{-1}|t|^{-(p-1)} = b \quad \text{uniformly on } x \in \partial\Omega, \text{ where } b > 0 \tag{1.4}$$

and p satisfies (1.3). Note that (1.4) implies (1.1) and (1.2).

Our main goal is to study the properties of the solutions of (P). We will prove that their boundedness is equivalent to the finiteness of their Morse indices.

There have been many works devoted to the relationship between the properties of solutions and Morse indices. The earliest and the most known result is due to Bahri and Lions [1], they studied the following problem:

$$-\Delta u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.5}$$

Under similar conditions on f, they proved that bounds on solutions of the above problem are equivalent to bounds on their Morse indices. Bahri and Lions needed this useful information when they apply topological methods to establish existence and multiplicity result for (1.5) (see for instance [2,3]). To get the a priori estimate, they used a classical blow up argument which leads to treat those systems:

$$\begin{cases} -\Delta u = |u|^{q-1}u \quad \text{in } \mathbb{R}^N \\ u \in \mathcal{C}_b^2(\mathbb{R}^N), \quad i(u) < \infty \end{cases}$$

and

$$\begin{cases} -\Delta u = |u|^{q-1}u \text{ in } \mathbb{R}_+^N, \quad u = 0 \text{ on } \partial\mathbb{R}_+^N \\ u \in \mathcal{C}_b^2(\mathbb{R}_+^N), \quad i(u) < \infty \end{cases}$$

where $i(u)$ denotes the Morse index of $u, \mathcal{C}_b^2$ denotes the set of bounded and $\mathcal{C}^2$ functions, $1 < q < \frac{N+2}{N-2}$ when $N \ge 3$ and $q \in (1,\infty)$ if $N = 2$. They proved that the problems below have only the trivial solution $u \equiv 0$. This result extended the nonexistence result of positive solutions in [4] to finite Morse index solution.

Later, Harrabi et al. [5,6] generalize the work of Bahri and Lions to other nonlinearities. Recently, Harrabi et al. [7] studied the corresponding Neumann boundary value problem and X. Yu studied the mixed boundary problems in [8]. Concerning the supercritical case, one can refer to the following works [9–11].