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Regularity criteria for the Navier–Stokes equations based on one or two items of the velocity gradient

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ABSTRACT

We study the regularity criteria for the incompressible Navier–Stokes equations based on either one item of the velocity gradient, $\partial_1 u_3$ or $\partial_3 u_3$ or on two items of the velocity gradient, $\partial_2 u_3$, $\partial_3 u_3$, or $\partial_1 u_3$, $\partial_2 u_3$. We improve and/or extend several results from the literature by the use of two versions of the generalized Troisi inequality.

1. Introduction

We consider the Navier–Stokes equations in the entire three-dimensional space, i.e.

$$\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla p = f \quad \text{in } \mathbb{R}^3 \times (0, \infty), \tag{1}$$

$$\nabla \cdot u = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty), \tag{2}$$

$$u|_{t=0} = u_0, \tag{3}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote the unknown velocity and pressure, $\nu > 0$ is the kinematic viscosity, f is the external force and $u_0 = u_0(x)$ is the initial velocity vector field. In what follows, we put, without loss of generality, $\nu = 1$, and $f \equiv 0$ for simplicity. As is well known, the system (1)–(3) models the flow of a viscous incompressible fluid.

It was proved a long time ago (see [1]) that for $u_0 \in L^2_\sigma$ (solenoidal functions from L^2) the problem (1)–(3) possesses at least one global weak solution u satisfying the energy inequality $\|u(t)\|_2^2/2 + \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \leq \|u_0\|_2^2/2$ for every $t \geq 0$ (see [1] or [2]). Such solutions are called the Leray solutions. If $u_0 \in W^{1,2}_\sigma$ (solenoidal functions from the standard Sobolev space $W^{1,2}$) then the Leray solutions are regular on some (possibly small) time interval $(0, \delta]$, $\delta > 0$. It means that $\nabla u \in L^\infty((0,\delta); L^2)$, $u \in L^2(0,\delta; W^{2,2})$ and (subsequently)

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$u \in C^\infty((0,\delta) \times \mathbb{R}^3)$ (see [2]). As is well known, the mathematical theory of the system (1)–(3) is not complete. It is a classical question to ask whether or not the Leray solutions are regular on an arbitrary interval $(0,T]$, $T > 0$. This fundamental problem has not yet been solved and seems to be beyond the scope of the present techniques. Nevertheless, there exist plenty of results in the literature showing that the answer to this problem is positive if some additional conditions are imposed on the Leray solutions. Mention here specifically the following classical regularity result known as the Prodi–Serrin conditions (see [3] and [4] for $q > 3$ and [5] for $q = 3$): a Leray solution u with the initial condition $u_0 \in W^{1,2}_\sigma$ is regular on $(0,T]$, if $u \in L^t(0,T;L^q)$, where $q \in [3,\infty]$, $t \in [2,\infty]$ and

$$\frac{2}{t} + \frac{3}{q} = 1. \tag{4}$$

It is well known that if u and p solve the system (1)–(2) then the same is true for the rescaled functions u_λ, p_λ, $\lambda > 0$, defined as

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \qquad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t).$$

The spaces from (4) are called critical since their norms are invariant with regard to the presented scaling, that is $\|u\|_{L^t(0,T;L^q)} = \|u_\lambda\|_{L^t(0,T/\lambda^2;L^q)}$ for every $\lambda > 0$. In this sense the Prodi–Serrin conditions are optimal.

An analogical situation occurs for ∇u: it was proved in [6] that u with the initial condition $u_0 \in W^{1,2}_\sigma$ is regular on $(0,T]$ if $\nabla u \in L^t(0,T;L^q)$, where $q \in (3/2,\infty)$ and

$$\frac{2}{t} + \frac{3}{q} = 2. \tag{5}$$

This result is also optimal since the spaces from (5) are scale invariant considering ∇u instead of u: $\|\nabla u\|_{L^t(0,T;L^q)} = \|\nabla u_\lambda\|_{L^t(0,T/\lambda^2;L^q)}$.

The above mentioned criteria (4) and (5) are based on the entire velocity vector or on the entire velocity gradient. In the last two decades many authors have studied the regularity criteria where the additional conditions were imposed only on some velocity components or some items of the velocity gradient, see for example [7–16] and [17]. The results in [7,10–12,15] are optimal, at least for some range of q. It is not the case for criteria we are going to study in the present paper. We will focus on criteria based only on (1) one item of the velocity gradient, $\partial_1 u_3$ or $\partial_3 u_3$, or on (2) two items of the velocity gradient, $\partial_1 u_3, \partial_2 u_3$ or $\partial_2 u_3, \partial_3 u_3$. These criteria have been studied in several recent papers, see [18,8,19–21]. Let us start with the presentation of the known results. At first, discuss the regularity conditions imposed on one item of the velocity gradient. It was proved in [18] that u is regular on $(0,T]$ if $\partial_3 u_3 \in L^t(0,T;L^q)$, where $q \in (15/4,\infty]$ and

$$\frac{2}{t} + \frac{3}{q} < \frac{4}{5}.$$

In [8] the authors came with a key idea: estimating the integral form of the convective form, $\int u\nabla u\phi$, they were able to conveniently excerpt the term $\partial_1 u_3$ ($\partial_3 u_3$) and so directly apply the condition imposed on this term. It leads to a significant improvement of the result from [18]: u is regular on $(0,T]$ if either $\partial_1 u_3 \in L^t(0,T;L^q)$, where $q > 3$, $t \in [1,\infty)$ and

$$\frac{2}{t} + \frac{3}{q} = \frac{q+3}{2q}, \tag{6}$$

or $\partial_3 u_3 \in L^t(0,T;L^q)$, where $q > 2$, $t \in [1,\infty)$ and

$$\frac{2}{t} + \frac{3}{q} = \frac{3(q+2)}{4q}. \tag{7}$$