Nonlinear Analysis: Real World Applications 38 (2017) 146–170

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Nontrivial solutions of some fractional problems

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A R T I C L E I N F O

Article history:
Received 26 September 2016
Received in revised form 28 February 2017
Accepted 24 April 2017

Keywords:
Integrodifferential operators
Fractional Laplacian
Fountain Theorem

A B S T R A C T

In this paper, we study the question of the existence of infinitely many weak solutions for nonlocal equations of fractional Laplacian type with homogeneous Dirichlet boundary data, in presence of a superlinear term. Starting from the well-known Ambrosetti–Rabinowitz condition, we consider different growth assumptions on the nonlinearity, all of superlinear type. Furthermore, we give an extension of Ambrosetti–Rabinowitz condition, a non-Ambrosetti–Rabinowitz condition and apply to study the fractional Laplacian equation. We obtain some different existence results in this setting by using Fountain Theorem. Our results are extension of some problems studied by Bisci et al. (2016) and Binlin et al. (2015).

1. Introduction and main results

Recently, nonlocal fractional problems have been appearing in the literature in many different contexts, both in the pure mathematical research and in concrete real-world application. Indeed, fractional and nonlocal operators appear in many diverse fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conversion laws, ultra-relativistic of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surface, materials science and water waves. In this paper, we consider the existence of infinitely many solutions of two following problems:

$$\begin{cases} \mathcal{L}_K u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{1.1}$$

and

$$\begin{cases} \mathcal{L}_K u - \lambda u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.2}$$

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http://dx.doi.org/10.1016/j.nonrwa.2017.04.009

When $\lambda = 0$, the problem (1.2) gives the problem (1.1). The Ω is an open bounded subset of $\mathbb{R}^n$ with continuous boundary $\partial\Omega, n > 2s, s \in (0,1)$, then term f satisfies the different superlinear conditions, and $\mathcal{L}_K$ is the integrodifferential operator which is defined as follows:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^n, \tag{1.3}$$

where the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is such that

$$mK \in L^1(\mathbb{R}^n), \quad \text{where } m(x) = \min\{|x|^2, 1\}, \tag{1.4}$$

$$K(x) = K(-x) \quad \text{for all } x \in \mathbb{R}^n \tag{1.5}$$

and there exists $\theta > 0$ such that

$$K(x) \geq \theta |x|^{-(n+2s)} \tag{1.6}$$

for any $x \in \mathbb{R}^n \setminus \{0\}$. A model for K is given by the singular kernel $K(x) = |x|^{-(n+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$, defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \quad x \in \mathbb{R}^n.$$

Under superlinear and subcritical conditions on f, Servadei–Valdinoci [1,2] proved the existence of a nontrivial solution of (1.1) and (1.2) by using the Mountain Pass Theorem [3] and Linking Theorem [4]. Motivated by these works, in this paper we will study the existence of infinitely solutions of (1.1) and (1.2), as an application of Fountain Theorem due to Bartsch [5]. Our results are extension of some problems studied by Bisci–Repovs–Servadei [6] and Binlin–Bisci–Servadei [7].

1.1. Variational formulation of the problem

In order to study problem (1.1), we shall consider their weak formulation, given by

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dxdy = \int_\Omega f(x, u(x))\varphi(x)dx, \quad \varphi \in X_0, \\ u \in X_0 \end{cases} \tag{1.7}$$

which represents the Euler–Lagrange equation of energy functional $\mathcal{I}_K : X_0 \to \mathbb{R}$ defined as

$$\mathcal{I}_K(u) := \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y)dxdy - \int_\Omega F(x, u(x))dx, \tag{1.8}$$

where the function F is the primitive of f with respect to the second variable, that is,

$$F(x,t) = \int_0^t f(x,\tau)d\tau. \tag{1.9}$$

Similarly, in order to study problem (1.2), we shall consider their weak formulation, given by

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dxdy - \lambda \int_\Omega u(x)\varphi(x)dx = \int_\Omega f(x, u(x))\varphi(x)dx, \\ u \in X_0, \qquad \varphi \in X_0 \end{cases} \tag{1.10}$$

which represents the Euler–Lagrange equation of energy functional $\mathcal{J}_{K,\lambda} : X_0 \to \mathbb{R}$ defined as

$$\mathcal{J}_{K,\lambda}(u) := \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y)dxdy - \frac{\lambda}{2} \int_\Omega |u(x)|^2 dx - \int_\Omega F(x, u(x))dx. \tag{1.11}$$

Here, the space X_0 is defined by

$$X_0 := \{g \in X : g = 0 \text{ in } x \in \mathbb{R}^n \setminus \Omega\},$$