Nonlinear Analysis: Real World Applications 38 (2017) 222–244

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Existence of stationary turbulent flows with variable positive vortex intensity

F. De Marchis[a,*], T. Ricciardi[b]

[a] *Dipartimento di Matematica, Università deli Studi di Roma Sapienza, P.le Aldo Moro 5, 00185 Roma, Italy*

[b] *Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli Federico II, Via Cintia, Monte S. Angelo, 80126 Napoli, Italy*

ARTICLE INFO

Article history:
Received 26 July 2016
Received in revised form 22 February 2017
Accepted 30 April 2017

Keywords:
Mean field equation
Min–max solutions
Turbulent Euler flow

ABSTRACT

We prove the existence of stationary turbulent flows with arbitrary positive vortex circulation on non-simply connected domains. Our construction yields solutions for all real values of the inverse temperature with the exception of a quantized set, for which blow-up phenomena may occur. Our results complete the analysis initiated in Ricciardi and Zecca (2016).

1. Introduction and main results

Motivated by the statistical mechanics description of turbulent 2D Euler flows in equilibrium, we are interested in the existence of solutions to the following problem:

$$\begin{cases} -\Delta u = \lambda \dfrac{\int_{[0,1]} \alpha e^{\alpha u}\,\mathcal{P}(d\alpha)}{\iint_{[0,1]\times\Omega} e^{\alpha u}\,\mathcal{P}(d\alpha)dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (*)_\lambda$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $\lambda > 0$ is a constant and $\mathcal{P} \in \mathcal{M}([0,1])$ is a Borel probability measure. Problem $(*)_\lambda$ was derived by Neri [1] within Onsager's pioneering framework [2], with the aim of including the case of variable vortex intensities. More precisely, in [1] the following mean field equation is

---

* Corresponding author.
E-mail addresses: demarchis@mat.uniroma1.it (F. De Marchis), tonricci@unina.it (T. Ricciardi).

http://dx.doi.org/10.1016/j.nonrwa.2017.04.013

derived:

$$\begin{cases} -\Delta v = \dfrac{\int_{[-1,1]} re^{-\beta r v}\,\mathcal{P}(dr)}{\iint_{[-1,1]\times\Omega} e^{-\beta r v}\,\mathcal{P}(dr)dx} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, v is the mean field stream function of an incompressible turbulent Euler flow, the Borel probability measure $\mathcal{P} \in \mathcal{M}([-1,1])$ describes the vortex intensity distribution and $\beta \in \mathbb{R}$ is a constant related to the inverse temperature. The mean field equation (1.1) is derived from the classical Kirchhoff–Routh Hamiltonian for the N-point vortices system:

$$H^N(r_1,\dots,r_N,x_1,\dots,x_N) = \sum_{i\neq j} r_i r_j G(x_i,x_j) + \sum_{i=1}^N r_i^2 H(x_i,x_i),$$

in the limit $N \to \infty$, under the *stochastic* assumption that the r_i's are independent identically distributed random variables with distribution $\mathcal{P}$. In the above formula, for $x,y \in \Omega$, $x \neq y$, $G(x,y)$ denotes the Green's function defined by

$$\begin{cases} -\Delta G(\cdot,y) = \delta_y & \text{in } \Omega \\ G(\cdot,y) = 0 & \text{on } \partial\Omega \end{cases}$$

and $H(x,y)$ denotes the regular part of G, i.e.

$$H(x,y) = G(x,y) + \frac{1}{2\pi}\log|x-y|. \tag{1.2}$$

Setting $u := -\beta v$ and $\lambda = -\beta$, and assuming that

$$\operatorname{supp}\mathcal{P} \subset [0,1], \tag{1.3}$$

problem (1.1) takes the form $(*)_\lambda$. We recall that

$$\operatorname{supp}\mathcal{P} := \{\alpha \in [-1,1] : \ \mathcal{P}(N) > 0 \text{ for any open neighborhood } N \text{ of } \alpha\}.$$

Assumption (1.3) corresponds to the case of physical interest where all vorticities have the same orientation.

We observe that without loss of generality we may assume

$$1 \in \operatorname{supp}\mathcal{P}. \tag{1.4}$$

Indeed, suppose that $\sup\operatorname{supp}\mathcal{P} = \bar\alpha \in (0,1)$. Then, $(*)_\lambda$ is equivalent to

$$\begin{cases} -\Delta u = \lambda\dfrac{\int_{[0,\bar\alpha]} \alpha e^{\alpha u}\,\mathcal{P}(d\alpha)}{\iint_{[0,\bar\alpha]\times\Omega} e^{\alpha u}\,\mathcal{P}(d\alpha)dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the change of variables $\alpha = \alpha'\bar\alpha$, $\bar{\mathcal{P}}(A) = \mathcal{P}(\bar\alpha A)$ for all Borel sets $A \subset [0,1]$, and setting $\bar u = \bar\alpha u$, we find that $\bar u$ satisfies

$$\begin{cases} -\Delta \bar u = \bar\alpha^2\lambda\dfrac{\int_{[0,1]} \alpha' e^{\alpha'\bar u}\,\bar{\mathcal{P}}(d\alpha')}{\iint_{[0,1]\times\Omega} e^{\alpha'\bar u}\,\bar{\mathcal{P}}(d\alpha')dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which is nothing but $(*)_{\bar\alpha^2\lambda}$, with $\bar{\mathcal{P}}$ satisfying (1.4). Henceforth, we always assume (1.4).

When $\mathcal{P}(d\alpha) = \delta_1(d\alpha)$ problem $(*)_\lambda$ reduces to the *standard* mean field problem

$$\begin{cases} -\Delta u = \lambda\dfrac{e^u}{\int_\Omega e^u\,dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$