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Proper spaces for the asymptotic convergence of solutions of porous medium equation[☆]

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ABSTRACT

In this paper, we consider the problem that a weighted L^∞ space $W_\vartheta(\mathbb{R}^N)$ is proper or not for the asymptotic convergence of solutions of the porous medium equation. We find that there exists a critical exponent $\vartheta = \sigma$ of the proper spaces for the asymptotic problem proposed by Alikakos and Rostamian (1984).

1. Introduction

In this paper, we study the Cauchy problem of the porous medium equation

$$\frac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $m > 1$ and $u_0 \in W_\sigma(\mathbb{R}^N) \equiv \{\varphi \in L^1_{\text{loc}}(\mathbb{R}^N);\ |x|^\sigma \varphi(x) \in L^\infty(\mathbb{R}^N)\}$ with the norm $\|\varphi(x)\|_{W_\sigma(\mathbb{R}^N)} = \|\,|x|^\sigma\varphi(x)\|_{L^\infty(\mathbb{R}^N)}$.

The solutions $u(x,t)$ of evolution equations usually decay to 0 or grow to ∞ as $t\to\infty$, see [1]. To describe the asymptotic behavior of solutions, the classical method is to apply a proper time scale on the solutions, and study the large time behavior of the rescaled solutions $t^\mu u(t^\beta x, t)$ in one proper space [2–4]. Consider the problem (1.1)–(1.2). A classical result about the asymptotic behavior of the solutions with integrable

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data had been given by Friedman and Kamin in 1980 [5], who showed that if $0 \le u_0 \in L^1(\mathbb{R}^N)$, then

$$\lim_{t\to\infty} t^{\frac{N}{N(m-1)+2}} |u(t^{\frac{1}{N(m-1)+2}}x,t) - U_M(t^{\frac{1}{N(m-1)+2}}x,t)| \to 0 \quad \text{in } L^\infty_{\text{loc}}(\mathbb{R}^N), \tag{1.3}$$

where U_M is the Barenblatt solution with the same mass M as that of u_0. The first proof of (1.3) in one dimension appeared in 1973 [6]. In 1988, Kamin and Vázquez found in a celebrated paper [7] that (1.3) also holds even if the space $L^\infty_{\text{loc}}(\mathbb{R}^N)$ is substituted by $L^\infty(\mathbb{R}^N)$, see also [8]. In 1984, Alikakos and Rostamian [9] revealed that if the initial data $0 \le u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$ is such that

$$\lim_{|\lambda|\to\infty} |\lambda|^\sigma u_0(\lambda x) = A|x|^{-\sigma} \quad \text{in } \mathscr{D}'(\mathbb{R}^N) \tag{1.4}$$

with $-\frac{2}{m-1} < \sigma \le N$, then

$$t^{\frac{\sigma}{\sigma(m-1)+2}} |u(t^{\frac{1}{\sigma(m-1)+2}}x,t) - W(t^{\frac{1}{\sigma(m-1)+2}}x,t)| \xrightarrow{t\to\infty} 0 \tag{1.5}$$

in $L^\infty_{\text{loc}}(\mathbb{R}^N)$, where $W(x,t)$ is the solution of (1.1) with the initial data $W(x,0) = A|x|^{-\sigma}$. We found that if $0 < \sigma < N$, then (1.5) holds in $L^\infty(\mathbb{R}^N)$ [10]. Analogous asymptotic results of (1.5) in L^∞_{loc} for the porous medium equation with absorption had been obtained by Kamin and Peletier in 1986 [11]. By applying a proper time scale, some other interesting results about the asymptotic behavior of solutions of the porous medium equation in one proper space had been obtained, one can refer to [12–16].

To best understand the asymptotic behavior of solutions of Problem (1.1)–(1.2), we apply a space–time scale to the solutions, and research the large time behavior of the rescaled solutions $|x|^\vartheta t^\mu u(t^\beta x,t)$ in the space $L^\infty(\mathbb{R}^N)$. This method can be improved by studying the large time behavior of the rescaled solutions $t^\mu u(t^\beta x,t)$ in the space $W_\vartheta(\mathbb{R}^N)$. Our main concern here is to investigate the exponents ϑ where the spaces $W_\vartheta(\mathbb{R}^N)$ are proper or not for the asymptotic convergence (1.5). Observe that (1.5) holds in a space if and only if the following limit holds in the same space:

$$t^{\frac{\sigma}{\sigma(m-1)+2}} u(t^{\frac{1}{\sigma(m-1)+2}}x,t) \xrightarrow{t\to\infty} W(x,1), \tag{1.6}$$

see details in (4.7). We find that if the initial data $u_0 \in W^+_\sigma(\mathbb{R}^N) \equiv \{\varphi \in W_\sigma(\mathbb{R}^N);\ \varphi(x) \ge 0\}$ with $0 < \sigma < N$ satisfy (1.4), then (1.5) holds in $W_\vartheta(\mathbb{R}^N)$ with any $0 \le \vartheta < \sigma$, while there exists an initial datum $u_0 \in W^+_\sigma(\mathbb{R}^N)$ satisfies (1.4), the limit (1.6) cannot hold in $W_\vartheta(\mathbb{R}^N)$ with all $\vartheta > \sigma$. So for $0 < \sigma < N$, we reveal that for the asymptotic problem proposed by Alikakos and Rostamian [9], the spaces $W_\vartheta(\mathbb{R}^N)$ $(0 \le \vartheta < \sigma)$ are the proper spaces, but $W_\vartheta(\mathbb{R}^N)$ $(\vartheta > \sigma)$ are not. Therefore, $\vartheta = \sigma$ is the critical exponent for the spaces $W_\vartheta(\mathbb{R}^N)$ (the space scaled exponents).

Remark 1.1. If $\vartheta = 0$, then $W_0(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$. So for $0 < \sigma < N$, our results extend the works of Alikakos and Rostamian [9] from $L^\infty_{\text{loc}}(\mathbb{R}^N)$ to $W_\vartheta(\mathbb{R}^N)$ with $0 \le \vartheta < \sigma$.

The Banach space $W_\sigma(\mathbb{R}^N)$ first appeared in [17,18] to study the complicated asymptotic behavior of solutions of the heat equation. For more applications of $W_\sigma(\mathbb{R}^N)$, one can see [19–23].

Remark 1.2. For $0 < \sigma < N$, the problem that the critical exponent space $W_\vartheta(\mathbb{R}^N)$ $(\vartheta = \sigma)$ is a proper space or not has still remained unsettled.

The rest of this paper is organized as follows. In the next section, we introduce some concepts and estimates. Section 3 is devoted to studying the proper spaces for the asymptotic convergence of the solutions. In the last section we consider the improper spaces for the asymptotic convergence of the solutions.