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A quasi-static evolution generated by local energy minimizers for an elastic material with a cohesive interface

M. Negri[a,*], R. Scala[b]

[a] Department of Mathematics, University of Pavia, Via A. Ferrata 1, 27100 Pavia, Italy
[b] Faculdade de Ciencias da Universidade de Lisboa, Campo Grande, 1749-016 Lisboa, Portugal

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ABSTRACT

We deal with a model for an elastic material with a cohesive crack along a prescribed fracture set. We consider two n-dimensional elastic bodies and a cohesive law, on their common interface, with incompenetrability constraint and general loading–unloading regimes. We first provide a time-discrete evolution by means of local minimizers of the energy with respect to the L^2-norm of the crack opening displacement. The choice of this norm is due to technical reasons (the λ-convexity of the energy) and is in analogy with the classical approach in quasi-static brittle fracture, where the evolution of the system is condensed into the evolution of the crack. In the "time-continuous" limit we obtain a BV-evolution, in parametrized form, characterized by Karush–Kuhn–Tucker conditions, for the internal variable, equilibrium and energy identity.

1. Introduction

Starting from pioneering works in mechanics (see [1–3] for basic models) the analysis of cohesive zone models has obtained much attention in the recent years, especially due to the numerous applications in engineering. Analysis has been carried on by several authors under different settings and considering several mechanical aspects of the problem. Recently a common assumption relies in considering models where the fracture is confined to a prescribed interface between two elastic or visco-elastic bodies. This hypothesis is considered in many contributions ([4–13], and references therein) and is also assumed in the present work.

Let us spend some words on the model we consider: let $T > 0$, we consider a reference configuration $\Omega = (\Omega^+ \cup \Omega^-) \subset \mathbb{R}^n$ given by the union of two elastic bodies, separated by an interface $K = \partial\Omega^+ \cap \partial\Omega^-$ representing the crack. The displacement $u : [0, T] \times \Omega \to \mathbb{R}^n$ has linearized strain $\boldsymbol{\epsilon}(u)$ and signed opening displacement $[\![u]\!] = u^+ - u^-$ on K. A fundamental feature of our model is the presence of an internal variable $\xi : [0, T] \times K \to [0, +\infty]$ taking into account the history of the separation between the two bodies (see e.g. [14,15]); $\xi(x)$ represents the maximum separation that has taken place at the point $x \in K$ during

* Corresponding author.
E-mail addresses: matteo.negri@unipv.it (M. Negri), rscala@fc.ul.pt (R. Scala).

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the evolution. In particular $\xi(x) = 0$ means that no opening of the crack has still happened at x. As a consequence, the internal variable satisfies the irreversibility condition $\dot{\xi} \geq 0$ while the crack opening satisfies $|[\![u]\!]| \leq \xi$. In particular dissipation occurs only when $\dot{\xi} > 0$ and $|[\![u]\!]| = \xi$ (loading) while a different non-dissipative mechanical behaviour occurs when $\dot{\xi} = 0$ and $|[\![u]\!]| < \xi$.

We study a quasi-static evolution for the energy

$$F(u,\xi) = \int_{\Omega} W(\boldsymbol{\epsilon}(u))\,dx + \int_{K} \varphi([\![u]\!],\xi,\nu)\,d\mathcal{H}^{n-1},$$

where ν is the unit normal on K, W is the bulk elastic energy, and φ is the cohesive potential. For the precise technical and mechanical assumptions the reader is referred to Section 2; we point out that φ is λ-convex w.r.t. $[\![u]\!]$, non-decreasing w.r.t. ξ, and that it takes into account both infinitesimal incompenetrability and different loading–unloading regimes. We remark that in this representation the potential φ can be written also as $\varphi([\![u]\!],\xi,\nu) = \varphi_s([\![u]\!],\xi,\nu) + \varphi_d(\xi)$ where φ_s is the stored energy while φ_d is the dissipated energy; in particular, the system is rate-independent since dissipation, i.e. rate of dissipated energy, takes the form $\mathcal{D}(\xi(t),\dot{\xi}(t)) = \varphi_d'(\xi(t))\,\dot{\xi}(t)$. For simplicity, we consider the case where the quasi static evolution is governed only by a time-dependent Dirichlet boundary condition $u = g(t)$ for $t \in [0,T]$ on $\partial_D\Omega$, the Dirichlet part of the boundary, whereas the external forces acting on the bodies and external tractions on $\partial_N\Omega = \partial\Omega \setminus \partial_D\Omega$ are set equal to 0. Therefore, the equilibrium equations of the system will be "qualitatively" of the form

$$\begin{cases} \operatorname{div}\boldsymbol{\sigma}(t) = 0 & \text{on } \Omega,\\ \boldsymbol{\sigma}_\nu(t) = 0 & \text{on } \partial_N\Omega,\\ \boldsymbol{\sigma}_\nu(t) \in \partial_{[\![u]\!]}\varphi([\![u(t)]\!],\xi(t),\nu) & \text{on } K,\\ u(t) = g(t) & \text{on } \partial_D\Omega, \end{cases}$$

where $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon}(u)$ is the linear stress tensor, $\boldsymbol{\sigma}_\nu = \boldsymbol{\sigma}\nu$ is the normal tension, $\partial_{[\![u]\!]}\varphi$ denotes the subdifferential with respect to $[\![u]\!]$ (in particular $[\![u(t)]\!]\cdot\nu \geq 0$ on K). These equations will be given a rigorous and detailed formulation in Section 5. As for the flow rule, we have to specify which kind of quasi-static evolution we are interested in. More precisely, we look for a parametrized BV-evolution in the sense of [16]. In general, BV-evolutions for rate-independent problems are defined by vanishing viscosity, taking the limit of rate-dependent parabolic evolutions. In this work we follow a different approach, we endow the space of admissible configurations with a norm $\|\cdot\|$ and we generate the evolutions by means of a time-discrete scheme based on local minimizers of F (similar schemes can be found in [13,17,18]). Given the initial configuration u_0, ξ_0 at time $t_0 = 0$ and given $\Delta > 0$, the discrete evolution is defined by induction as follows:

- if $t_k < T$ and u_k is a local minimizer of $F(t_k,u,\xi_k)$ (in some neighbourhood $\|u - u_k\| \leq r$ for $r > 0$) then

$$\begin{cases} t_{k+1} = \min\{t_k + c\Delta, T\}\\ u_{k+1} = u_k,\\ \xi_{k+1} = \xi_k, \end{cases}$$

  (for a suitable choice of $c > 0$)
- if $t_k \leq T$ and u_k is not a local minimizer then

$$\begin{cases} t_{k+1} = t_k,\\ u_{k+1} \in \operatorname{argmin}\{F(t_k,u,\xi_k) : \|u - u_k\| \leq \Delta\},\\ \xi_{k+1} = \xi_k \vee |[\![u_{k+1}]\!]|. \end{cases}$$

Several comments are due. If $\|u_{k+1} - u_k\| < \Delta$ then u_{k+1} is a local minimizer of $F(t_k,\cdot,\xi_k)$ in a ball $B(u_{k+1},r)$, for r sufficiently smaller than Δ; on the contrary, if $\|u_{k+1} - u_k\| = \Delta$ then u_{k+1} is not necessarily a local minimizer of $F(t_k,\cdot,\xi_k)$ in a ball $B(u_{k+1},r)$. In particular, some indices k may not correspond to a local minimizer of $F(t_k,\cdot,\xi_k)$. Selecting the indices k_i such that u_{k_i} is a local minimizer at time $t_{k_i} = ic\Delta$