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Solution to the interior transmission problem using nodes on a subinterval as input data

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ABSTRACT

It is well known that a spherically symmetric wave speed problem in a bounded spherical region may be reduced, by means of Liouville transform, to the Sturm–Liouville problem $B(a;q)$ in $[0,1]$. We prove that a twin dense subset of the nodal set in a subinterval ($\subset [0,1]$) uniquely determines the potential q on the whole interval $[0,1]$ and the boundary parameter a. The method employed is to convert the inverse nodal problem into the inverse spectral problem with partial information given on the potential on a subinterval.

1. Introduction

The interior transmission problem is a non-self adjoint boundary-value problem for a pair of fields Ψ and Ψ_0 in a bounded and simply connected domain Ω_b of $\mathbb{R}^3$ with the sufficiently smooth boundary $\partial\Omega_b$. It was first stated in [1] and can be formulated [1–3] as

$$\begin{cases} \Delta\Psi + k^2 n(x)\Psi = 0, \quad x \in \Omega_b, \\ \Delta\Psi_0 + k^2\Psi_0 = 0, \quad x \in \Omega_b, \\ \Psi = \Psi_0, \qquad \dfrac{\partial\Psi}{\partial\mathbf{n}} = \dfrac{\partial\Psi_0}{\partial\mathbf{n}}, \quad x \in \partial\Omega_b, \end{cases} \tag{1.1}$$

where Ω_b is a ball of radius $b > 0$ centered at the origin and $n(x)$ is spherically symmetric ($n(x) = n(r)$, $r = |x|$), Δ denotes the Laplacian, k^2 is the spectral parameter, $\mathbf{n}$ represents the outward unit normal to the boundary $\partial\Omega_b$, and the positive quantity $n(x)$ corresponds to the square of the refractive index of the

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medium at location x in the electromagnetic case or the reciprocal of the square of the sound speed $v(x)$ in the acoustic case. This boundary value problem is called the interior transmission problem.

Our assumptions on $n(r)$ are that $n(r)$ is positive and $n(r) \in W_2^2[0,b], n(b) = 1, n'(b) = 0$. Under the above assumptions this inverse problem to recover $n(x)$ from some observable data, is equivalent to recovering the potential $q(x)$ from the spectrum or other observable data (see Appendix), of the following boundary value problem

$$B(a,q): \begin{cases} y''(x) + (\lambda - q(x))y(x) = 0, \quad 0 < x < 1, \\ y(0) = 0 = y(1)\cos\left(\sqrt{\lambda}a\right) - y'(1)\dfrac{\sin\left(\sqrt{\lambda}a\right)}{\sqrt{\lambda}}. \end{cases} \tag{1.2}$$

Here $\sqrt{\lambda}$ is the square root branch with $\mathrm{Im}(\sqrt{\lambda}) \geq 0$ and

$$q(x) = B^2\Big(\frac{n''(r)}{4n^2(r)} - \frac{5(n'(r))^2}{16n^3(r)}\Big), \quad x = \frac{1}{B}\int_0^r \sqrt{n(\zeta)}d\zeta,$$
$$\lambda = B^2k^2, \qquad a = \frac{b}{B}, \quad B = \int_0^b \sqrt{n(\zeta)}d\zeta.$$

Denote $\tilde{n}(x) := n(r)$. Then the function $\sqrt[4]{\tilde{n}(x)}$ satisfies the following Cauchy problem:

$$\left(\sqrt[4]{\tilde{n}(x)}\right)'' = q(x)\sqrt[4]{\tilde{n}(x)}, \quad 0 < x < 1,$$
$$\sqrt[4]{\tilde{n}}(1) - 1 = 0 = \left(\sqrt[4]{\tilde{n}}\right)'(1).$$

Thus, $q(x)$ uniquely determines $\tilde{n}(x)$, $0 < x < 1$. Again, from $x = \frac{1}{B}\int_0^r \sqrt{n(\zeta)}d\zeta$ we get

$$\frac{dr}{dx} = \frac{B}{\sqrt{\tilde{n}(x)}}$$

with initial condition

$$r(0) = 0.$$

From the uniqueness of the solution of the differential equation it follows that $\tilde{n}(x)$ uniquely determines $r(x)$. Therefore, the determination of the function $n(r)$ is equivalent to the determination of the function $q(x)$. In light of the above equivalency, to discuss the unique recovery problem of $n(x)$, we only need to consider the uniqueness of recovering q in (1.2) in terms of appropriately given data.

The values of k^2 for which (1.1) has a pair of nontrivial solutions Ψ and Ψ_0 are called transmission eigenvalues. This interior transmission problem arises in the inverse scattering theory in inhomogeneous media, where the goal is to determine the function $n(x)$ in Ω_b from an appropriate set of k^2-values and other observable data related to (1.1). Related problems have been considered by a number of authors, see, e.g., [1–17].

We are concerned with the problem $B(a;q)$. Denote

$$A = \max\{e^{\|q\|}, (\|q\|+1)e^{\|q\|}+1\}, \qquad i_0 = \left[\max\left\{1, A, \frac{25A|1-a|}{\pi^2}\right\}\right].$$

When $a \neq 1$ it is well known that for the index $n > i_0$ the real eigenvalue λ_n of the problem $B(a,q)$ has the form [14]:

$$\lambda_n = \frac{n^2\pi^2}{(1-a)^2} + \frac{1}{1-a}\int_0^1 q(x)dx - \frac{1}{1-a}\int_0^1 q(t)\cos\frac{2\pi nt}{1-a}dt + O\left(\frac{1}{n}\right). \tag{1.3}$$