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# Harmonic solutions of a dry friction system[☆]

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ABSTRACT

In this paper we study harmonic solutions of the second order differential inclusion $\ddot{x} + g(x, \dot{x}) + u(x)\mathrm{Sgn}\dot{x} \ni \phi(t)$, which models a mechanical system with dry friction, viscous damping and $T$-periodic external force. Under a weaker assumption than published works, we give sufficient conditions for the existence of harmonic solutions not only by the Degree Theory when $g(x, \dot{x})$ depends on $\dot{x}$ but also by the Schauder's Fixed Point Theorem when $g(x, \dot{x})$ is independent of $\dot{x}$. After obtaining the necessary and sufficient condition for the existence of constant solutions, we find the number of harmonic solutions based on the number of constant solutions when $u(x)$ is a positive constant.



## 1. Introduction

Dry friction usually exists in many mechanical experiments and attracts many researchers. Many mathematical models are upbuilt for such systems and interesting results about solutions are obtained including the boundedness (see, e.g., [1–3]) and periodicity (see, e.g., [2,4–9]). The simplest dry friction model is $\ddot{x} + x + \mathrm{sgn}\dot{x} = \phi(t)$ with $T$-periodic $\phi(t)$, which models the pendulum with dry friction (see [2]). Here $x$ is displacement from rest and $\dot{x}$ is velocity. Many results of this equation are given in [2,7,8] including the existence, uniqueness and stability of harmonic solutions. Since the restoring maybe nonlinear, the equation

$$\ddot{x} + g(x) + u\mathrm{sgn}\dot{x} = \phi(t) \tag{1.1}$$

is studied in [10], where $u$ is a positive constant and $g$ usually corresponds to the restoring. A sufficient condition for the existence of harmonic solutions of (1.1) is given by an approximation method in [10] as

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well as its uniqueness and stability. On the other hand, the mechanical system may relate to a vicious damping and $u$ is positive but may depend on $x$, i.e.,

$$\ddot{x} + f(\dot{x}) + g(x) + u(x)\mathrm{sgn}\dot{x} = \phi(t), \tag{1.2}$$

where $f$ usually corresponds to a vicious damping. By the classical Degree Theory (see [11]), a sufficient condition for the existence of harmonic solutions of (1.2) is given in [6, Example 13.3] (respectively [4, Theorem 1]) when $u(x)$ is (respectively is not) a constant function and the uniqueness is also investigated under some conditions as well as the stability. Both [4,6] require two important assumptions:

$$u(r) \le u_0 + u_1|r| \quad \text{for all } r \in \mathbb{R} \text{ and some } u_0, u_1 \ge 0 \tag{1.3}$$

and

$$\sup_{r\in\mathbb{R}} |g(r) - \eta r| < +\infty, \qquad \sup_{r\in\mathbb{R}} |f(r) - \delta r| < +\infty \quad \text{for some } \eta, \delta > 0. \tag{1.4}$$

Note that (1.3) always holds in [6] because $u$ is a positive constant. Later, (1.4) is weakened as

$$|g(r) - \eta r| \le l_0 + l_1|r|^{\alpha}, \qquad |f(r) - \delta r| \le k_0 + k_1|r|^{\beta} \quad \text{for all } r \in \mathbb{R} \text{ and some } \eta, \delta > 0 \tag{1.5}$$

in [9], where $l_0, l_1, k_0, k_1 \ge 0$ and $\alpha, \beta \in (0, 1)$.

As indicated in [12, Section 76] and [13, Section 50], the dry friction can be modeled as the form $u(x)(h(\dot{x}) + \mathrm{sgn}\dot{x})$ in some cases, where $h$ is a nonnegative bounded continuous function. Thus, we consider a more general equation

$$\ddot{x} + g(x, \dot{x}) + u(x)\mathrm{sgn}\dot{x} = \phi(t), \tag{1.6}$$

where $g(x, \dot{x})$ usually denotes viscous damping and restoring but in some cases includes a part of dry friction such as $u(x)h(\dot{x})$ as mentioned above. Here we always assume that (1.6) satisfies the existence and uniqueness condition of solutions, both $g : \mathbb{R}^2 \to \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}^+$ are continuous, $\phi(t)$ is a Lebesgue measurable, bounded and $T$-periodic external force.

There is an interesting phenomenon, so-called stick–slip-motions (see [2,4] for details), because of the dry friction. Thus, Eq. (1.6) is usually considered as a differential inclusion

$$\ddot{x} + g(x, \dot{x}) + u(x)\mathrm{Sgn}\dot{x} \ni \phi(t), \tag{1.7}$$

where $\mathrm{Sgn}z := z/|z|$ when $z \ne 0$ and $\mathrm{Sgn}0 := [-1, 1]$. As indicated in [2, p. 3] inclusion (1.7) is the same as Eq. (1.6) for $\dot{x} \ne 0$. Moreover, the solution of (1.6) is defined as the solution of (1.7), which is an absolutely continuous function and satisfies the inclusion almost everywhere (see [14]).

In this paper, we study the existence and number of harmonic solutions, i.e., $T$-periodic solutions, of (1.7) and give some remarks on their stability. In our results, we also need two assumptions: (1.3) and there exist constants $\eta, \delta \in (0, \infty), \alpha_1, \alpha_2 \in (0, 1], c_0, c_1, c_2 \in [0, \infty)$ such that

$$|g(r_1, r_2) - \eta r_1 - \delta r_2| \le c_0 + c_1|r_1|^{\alpha_1} + c_2|r_2|^{\alpha_2} \quad \text{for all } (r_1, r_2) \in \mathbb{R}^2. \tag{1.8}$$

Clearly, (1.8) can be used for a more general form than $f(\dot{x}) + g(x)$. On the other hand, (1.8) is a weaker requirement than (1.5) for the special form $f(\dot{x}) + g(x)$. In fact, (1.8) always holds by taking $c_0 := l_0 + k_0, c_1 := l_1, c_2 := k_1, \alpha_1 := \alpha, \alpha_2 := \beta$ and $\eta, \delta$ as given in (1.5) if (1.5) holds. Moreover, (1.8) allows either $\alpha_1$ or $\alpha_2$ to be 1, i.e.,

$$0 < \min\{\alpha_1, \alpha_2\} \le \max\{\alpha_1, \alpha_2\} = 1, \tag{1.9}$$

which is not allowed in (1.5). In Section 2, we introduce some preliminaries about Degree Theory for multi-valued mappings, which is an efficient tool in studying the existence of periodic solutions of differential inclusions. In Section 3, we give sufficient conditions for the existence of harmonic solutions of (1.7) satisfying