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Equilibrium analysis for a mass-conserving model in presence of cavitation

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ABSTRACT

We study the existence of equilibrium positions for the load problem in Lubrication Theory. The problem consists of two surfaces in relative motion separated by a small distance filled by a lubricant. The system is described by the modified Reynolds equation (Elrod–Adams model) which describes the behavior of the lubricant and an extra integral equation given the balance of forces. The balance of forces allows to obtain the unknown position of the surfaces, defined with one degree of freedom.

1. Introduction and problem setting

Lubrication is the process used in mechanical systems to carry the load between two surfaces in relative motion and close proximity. The narrow space between the surfaces is filled by the lubricant, its characteristics allow to avoid the direct contact between the surfaces and reduce the wear. In that case, when distance is strictly positive we say that the system is in Hydrodynamic regime. The force induced by the pressure of the fluid is developed by the relative motion of the surfaces and it depends on the geometry of the space filled by the lubricant.

For simplicity, we assume that the bottom surface is planar and moves with a constant horizontal translation velocity. The lubricant is assumed incompressible and the distance between the surfaces belongs to the range of admissible distances satisfying the thin-film hypothesis, therefore the pressure fluid does not depend on the vertical coordinate.

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Let us denote by Ω the two-dimensional domain in which the hydrodynamical contact occurs. We assume, for simplicity, that $\Omega =]0,1[^2$ and the boundary $\partial\Omega$ is split into two parts: Γ_0 (defined by $\{x_1 = 0\}$) and the rest of the boundary, denoted by $\partial\Omega - \Gamma_0$. We also assume that the fluid flux at Γ_0 is a given constant $\mu > 0$. Without loss of generality we may assume that the velocity of the bottom surface is oriented in the direction of the x_1-axis and its normalized value is equal to 1.

In order to take into account the cavitation in the fluid we introduce the so-called Elrod–Adams model in the stationary case. The problem consists of finding p (the pressure of the lubricant) and θ (the volume fraction occupied by the fluid), solution of the following system (see for example [1–3]):

$$\begin{cases} \nabla \cdot \left[h^3(x)\nabla p\right] = \dfrac{\partial(\theta h)}{\partial x_1} \quad x \in \Omega \\ \theta \in H(p), \quad p \geq 0 \\ p = 0 \quad x \in \partial\Omega - \Gamma_0 \\ h\theta - h^3 \dfrac{\partial p}{\partial x_1} = \mu \quad \text{on } \Gamma_0 \end{cases} \tag{1.1}$$

where h is the non-dimensional distance (the gap) between the surfaces and H is the Heaviside multivalued function defined by

$$H(p) = \begin{cases} 1 & p > 0 \\ [0,1] & p = 0 \\ 0 & p < 0. \end{cases}$$

Notice that H is a multivalued function and $\theta \in H(p)$ belongs to $L^\infty(\Omega)$.

In many lubricated systems, the position of the surface is unknown and h may present some degrees of freedom. We reduce our study to the case where h will be given up to one degree of freedom which is the vertical translation, which results as an equilibrium position between the hydrodynamic force $\int_\Omega p(x)\,dx$ and the known exterior force F (assumed constant) applied upon the upper surface. Then, we assume that h is defined as follows

$$h(x) = h_0(x_1) + a \tag{1.2}$$

where $a > 0$ accounts for the vertical translation and $h_0 : [0,1] \to [0,\infty[$ is a given regular non-negative function which represents the gap corresponding to $a = 0$ and defines the geometry of the space. For simplicity we assume that the surface is rigid (i.e. h_0 is independent of the forces applied), depends only on x_1 and is $C^1([0,1])$ function. We also suppose that $\min_{x_1 \in [0,1]} h_0(x_1) = 0$, which allows to say that a represents the minimum distance between the two surfaces.

We are interested in the equilibrium positions of the system, which are defined as the stationary solutions of the equation defined by the second Newton's Law. Then, for a given constant force F, the problem consists in finding $a > 0$ such that,

$$\int_\Omega p\,dx = F \tag{1.3}$$

with h is defined in (1.2) and (p,θ) is a solution of (1.1).

We can also formulate this equilibrium problem as an *inverse problem* for the system (1.1)–(1.2): Find a parameter $a > 0$ such that the integral over Ω of the solution p is equal to F.

The problem for a general $h_0 \in C^1(\Omega)$ presents a high complexity and non-existence of solutions may occur for particular shapes h_0 and some exterior forces. Therefore we should restrict the study to the case where the contact region in the limit case ($a = 0$) satisfies the following assumptions: