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Quasi-linear Venttsel' problems with nonlocal boundary conditions on fractal domains

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ABSTRACT

Let $\Omega \subseteq \mathbb{R}^2$ be an open domain with fractal boundary $\partial\Omega$. We define a proper, convex and lower semicontinuous functional on the space $\mathbb{X}^2(\Omega, \partial\Omega) := L^2(\Omega, dx) \times L^2(\partial\Omega, d\mu)$, and we characterize its subdifferential, which gives rise to nonlocal Venttsel' boundary conditions. Then we consider the associated nonlinear semigroup T_p generated by the opposite of the subdifferential, and we prove that the corresponding abstract Cauchy problem is uniquely solvable. We prove that the (unique) strong solution solves a quasi-linear parabolic Venttsel' problem with a nonlocal term on the boundary $\partial\Omega$ of Ω. Moreover, we study the properties of the nonlinear semigroup T_p and we prove that it is order-preserving, Markovian and ultracontractive. At the end, we turn our attention to the elliptic Venttsel' problem, and we show existence, uniqueness and global boundedness of weak solutions.

0. Introduction

In the present paper we consider elliptic and parabolic quasi-linear Venttsel' problems with a nonlocal term on the boundary in a two dimensional fractal domain. Our interest in studying Venttsel' problems in irregular bounded domains with a fractal boundary is due to the fact that several natural and industrial

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processes lead to the formation of rough surfaces or occur across them. Fractal boundaries and fractal layers may be of great interest for those applications in which the surface effects are enhanced with respect to the surrounding volume (see [1–3] for details and motivations).

From the mathematical point of view the parabolic problems are modeled by an abstract nonhomogeneous quasilinear Cauchy problem for a maximal, m-accretive, monotone operator $\mathcal{A}$ which is the subdifferential of a proper, lower semicontinuous and convex energy functional (see (2.2)) which contains in particular a nonlocal term and a fractal nonlinear energy form.

To our knowledge this is the first example in which a nonlinear fractal local energy appears also in the boundary condition. When such energy term is associated with the tangential gradient at the boundary (for instance, on bounded Lipschitz domains), this problem has been addressed in [4]; however, this is not the case that occurs in this paper, as the notion of tangential gradient is not well defined on a fractal domain. To overcome this difficulty we use the notion of p-Lagrangians and hence of p-forms investigated in [5]. In the case considered here, we will combine some techniques and tools used previously in [6] for nonlocal Robin problems with those used in [7] for the Venttsel' linear problem to deal a nonlinear nonlocal evolution problem with a Venttsel boundary condition in domain with a fractal boundary. We give the characterization of the subdifferential, which in turn allows to prove that the solution of the abstract problem solves, in a suitable weak sense, the quasi-linear nonlocal Venttsel' problem. A key tool is Green's formula for fractal domains previously proved in [8,9]. We remark that we state the Venttsel' boundary condition in a suitable weak form since we do not have enough regularity information for traces of functions to the boundary of the domain; this lack of regularity already appears in the local linear case (see Theorem 6.1 in [7]).

Moreover in this paper we study the nonlinear semigroup generated by $-\mathcal{A}$ (see Theorem 3.1) and we prove that is order-preserving, Markovian and non-expansive. We also prove a nonlinear ultracontractivity property. In the study of these properties, we follow an approach similar to the one employed in [4,10] which in turn were motivated by the arguments introduced before in [11] for the linear case, and in [12,13] for quasi-linear equations with Dirichlet boundary conditions.

The presence of the nonlinear energy on the fractal boundary requires, when proving ultracontractivity, delicate estimates (see Lemma 3.4 in Section 3.2). We also investigate the corresponding inhomogeneous elliptic boundary value problem with nonlocal Venttsel' type boundary conditions on the given fractal domain. Under suitable assumptions on the data, we show the existence and uniqueness of solution in the weak sense. Moreover, a priori estimates for such solution are provided, and in particular, one obtains that weak solutions of the corresponding elliptic Venttsel' problem are globally bounded.

The literature on linear heat equation with Venttsel' boundary condition in smooth domains, say, at least bounded Lipschitz domains is large; we refer to the papers of [14–17] and to the references listed in. The case with fractal boundary is more recent (e.g. [18,7,9,8,19,20]). Venttsel' boundary conditions (also known as Wentzell boundary conditions) arise in many applications, such as phase-transition phenomena, fluid diffusion, and heat flow subject to nonlinear cooling on the boundary, suspended transport energy, fermentation, population dynamics, and climatology, among many others (see [21–26] and the references therein). The case of nonlinear parabolic operators involving a p-Laplace operator and generalized boundary conditions for Lipschitz domains is considered in [27], while the case of more general domains such as $W^{1,p}$-extension domain (see [28] for the definition of a $W^{1,p}$-extension domain) with nonlocal Robin boundary conditions are studied in [29,6] (and in [30] for the case of linear general operators with nonlocal boundary conditions in bounded Lipschitz domains). The well-posedness of the parabolic problem with dynamical boundary conditions on non-smooth and fractal domains has been recently investigated in [31,4].

The plan of the paper is the following. In Section 1 we introduce the snowflake domain $F := \partial\Omega$, we recall its properties and the construction of the p-energy forms on the Koch curve, and the main properties of the associated p-Lagrangian $\mathcal{L}^{(p)}$. In Section 2 we recall the definition of the main functional spaces involved and we introduce the functional Φ_p, we prove its properties (see Proposition 2.3) and we give a characterization of