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Delayed stability switches in singularly perturbed predator–prey models

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ABSTRACT

In this paper we provide an elementary proof of the existence of canard solutions for a class of singularly perturbed planar systems in which there occurs a transcritical bifurcation of the quasi steady states. The proof uses the one-dimensional result proved by V.F. Butuzov, N.N. Nefedov and K.R. Schneider, and an appropriate monotonicity assumption on the vector field. The result is applied to identify all possible predator–prey models with quadratic vector fields allowing for the existence of canard solutions.

1. Introduction

By multiple scale models we understand models of interlinked processes that occur at vastly different rates. In many cases the coexistence of such processes in the model is manifested by the presence of a small parameter that expresses the ratio of their characteristic times. Their mathematical modelling often leads to singularly perturbed systems of equations of the form

$$\begin{aligned} \mathbf{x}' &= \mathbf{f}(t, \mathbf{x}, \mathbf{y}, \epsilon), \qquad \mathbf{x}(0) = \mathring{\mathbf{x}}, \\ \epsilon \mathbf{y}' &= \mathbf{g}(t, \mathbf{x}, \mathbf{y}, \epsilon), \qquad \mathbf{y}(0) = \mathring{\mathbf{y}}, \end{aligned} \tag{1.1}$$

where $\mathbf{f}$ and $\mathbf{g}$ are sufficiently regular functions from an open subset of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+$ to, respectively, $\mathbb{R}^n$ and $\mathbb{R}^m$, for some $n, m \in \mathbb{N}$. It is of interest to determine the behaviour of solutions to (1.1) as $\epsilon \to 0$

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and, in particular, to show that they converge to the solutions of the *degenerate system*, obtained from (1.1) by letting $\epsilon = 0$. There are several reasons for this. First, taking such a limit in some sense 'incorporates' fast processes, described by the second equation of (1.1), into the slow dynamics, represented by the first one. Hence it links models acting at different time scales and often leads to new descriptions of natural processes, see e.g. [1]. Second, letting formally $\epsilon = 0$ in (1.1) yields a lower order system, whose solutions in many cases offer an approximation to the solution of (1.1) that retains the main dynamical features of the latter but can be obtained with less computational effort. In other words, often the qualitative properties of the solutions to (1.1) with $\epsilon = 0$ can be 'lifted' to $\epsilon > 0$ to provide a good description of dynamics of (1.1).

The first systematic analysis of problems of the form (1.1) was presented by A.N. Tikhonov in the 40' and this theory, with corrections due to F. Hoppenstead, can be found in e.g. [1–3]. Later, a parallel theory based on the centre manifold theory was given by F. Fenichel [4], though possibly the full reconciliation of these two theories only appeared in [5]. To introduce the main topic of this paper one should understand the main features of either theory and, since our work is more related to the Tikhonov approach, we shall focus on presenting the basics of the latter using the terminology of [1,6] that essentially is based on [3].

Let $\bar{\mathbf{y}}(t, \mathbf{x})$ be the solution to the equation

$$0 = \mathbf{g}(t, \mathbf{x}, \mathbf{y}, 0), \tag{1.2}$$

called the *quasi steady state*, and $\bar{\mathbf{x}}(t)$ be the solution to the *reduced equation*

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \bar{\mathbf{y}}(t, \mathbf{x}), 0), \qquad \mathbf{x}(0) = \mathring{\mathbf{x}}. \tag{1.3}$$

We assume that $\bar{\mathbf{y}}$ is an isolated solution to (1.2) in some set $[0, T] \times \bar{\mathcal{U}}$ and that it is an asymptotically stable equilibrium of the *auxiliary equation*

$$\frac{d\tilde{\mathbf{y}}}{d\tau} = \mathbf{g}(t, \mathbf{x}, \tilde{\mathbf{y}}, 0), \tag{1.4}$$

where here $(t, \mathbf{x})$ are treated as parameters, and that this stability is uniform with respect to $(t, \mathbf{x}) \in [0, T] \times \bar{\mathcal{U}}$, see [1, p. 81]. Further, assume that $\bar{\mathbf{x}}(t) \in \mathcal{U}$ for $t \in [0, T]$ provided $\mathring{\mathbf{x}} \in \bar{\mathcal{U}}$ and that $\mathring{\mathbf{y}}$ is in the basin of attraction of the equilibrium $\bar{\mathbf{y}}(0, \mathring{\mathbf{x}})$ of the *initial layer equation*

$$\frac{d\hat{\mathbf{y}}}{d\tau} = \mathbf{g}(0, \mathring{\mathbf{x}}, \hat{\mathbf{y}}, 0). \tag{1.5}$$

Theorem 1.1. *Let the above assumptions be satisfied. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in]\,0, \varepsilon_0]$ there exists a unique solution $(\mathbf{x}_\varepsilon(t), \mathbf{y}_\varepsilon(t))$ of (1.1) on $[0, T]$ and*

$$\begin{aligned} \lim_{\varepsilon \to 0} \mathbf{x}_\varepsilon(t) &= \bar{\mathbf{x}}(t), \quad t \in [0, T], \\ \lim_{\varepsilon \to 0} \mathbf{y}_\varepsilon(t) &= \bar{\mathbf{y}}(t), \quad t \in]\,0, T], \end{aligned} \tag{1.6}$$

where $\bar{\mathbf{x}}(t)$ is the solution of (1.3) and $\bar{\mathbf{y}}(t) = \bar{\mathbf{y}}(t, \bar{\mathbf{x}}(t))$ is the solution of (1.2).

We emphasize that the main condition for the validity of the Tikhonov theorem is that the quasi steady states be isolated and attractive; the latter in the language of dynamical systems is referred to as hyperbolicity, see e.g. [7]. In applications, however, we often encounter the situation when either the quasi steady state ceases to be hyperbolic along some submanifold (a fold singularity), or two (or more) quasi steady states intersect. The latter typically involves the so called 'exchange of stabilities' as in the transcritical bifurcation: the branches of the quasi steady states change from being attractive to being repelling (or conversely) across the intersection. The assumptions of the Tikhonov theorem fail to hold in the neighbourhood of the intersection