Nonlinear Analysis: Real World Applications 35 (2017) 379–396

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Regularity criteria for the Navier–Stokes equations based on one component of velocity

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ARTICLE INFO

Article history:
Received 21 June 2016
Received in revised form 30 October 2016
Accepted 16 November 2016

Keywords:
Navier–Stokes equations
Regularity of solutions
Regularity criteria
Anisotropic Lebesgue spaces

ABSTRACT

We study the regularity criteria for the incompressible Navier–Stokes equations in the whole space $\mathbb{R}^3$ based on one velocity component, namely u_3, ∇u_3 and $\nabla^2 u_3$. We use a generalization of the Troisi inequality and anisotropic Lebesgue spaces and prove, for example, that the condition $\nabla u_3 \in L^\beta(0,T;L^p)$, where $2/\beta + 3/p = 7/4 + 1/(2p)$ and $p \in (2,\infty]$, yields the regularity of u on $(0,T]$.

1. Introduction

We consider the Navier–Stokes equations in the full three-dimensional space, i.e.

$$\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla p = f \quad \text{in } \mathbb{R}^3 \times (0,\infty), \tag{1}$$

$$\nabla \cdot u = 0 \quad \text{in } \mathbb{R}^3 \times (0,\infty), \tag{2}$$

$$u|_{t=0} = u_0, \tag{3}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote the unknown velocity and pressure, $\nu > 0$ is the kinematic viscosity, f is the external force and $u_0 = u_0(x) = (u_{01}(x), u_{02}(x), u_{03}(x))$ is the initial velocity. In what follows, we put, without loss of generality, $\nu = 1$, and $f \equiv 0$ for simplicity. As is well known, the system (1)–(3) models the flow of a viscous incompressible fluid.

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http://dx.doi.org/10.1016/j.nonrwa.2016.11.005
1468-1218/

It was proved a long time ago (see [1]) that for $u_0 \in L^2_\sigma$ (solenoidal functions from L^2) the problem (1)–(3) possesses at least one global weak solution u satisfying the energy inequality $\|u(t)\|_2^2/2 + \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \leq \|u_0\|_2^2/2$ for every $t \geq 0$ (see [1] or [2]). Such solutions are called Leray solutions. If $u_0 \in W^{1,2}_\sigma$ (solenoidal functions from the standard Sobolev space $W^{1,2}$) then Leray solutions are regular on some (possibly small) time interval. The mathematical theory of the system (1)–(3) is, however, far from complete. It is a classical question to ask whether or not these solutions are regular on an arbitrary interval $(0,T], T > 0$, i.e. whether or not $\nabla u \in L^\infty((0,T);L^2), u \in L^2(0,T;W^{2,2})$ and (subsequently) $u \in C^\infty((0,T) \times \mathbb{R}^3)$ (see [2]). This renowned problem has not yet been solved and seems to be beyond the scope of the present techniques. Nevertheless, there exist many criteria in the literature ensuring the positive answer — see for example [3–16] and the references cited there. Mention here specifically the following classical regularity result known as the Prodi–Serrin conditions (see [17,18] for $p > 3$ and [7] for $p = 3$): a Leray solution u with the initial condition from $W^{1,2}_\sigma$ is regular on $(0,T]$ if

$$u \in L^\beta(0,T;L^p), \qquad 2/\beta + 3/p = 1, \quad p \in [3,\infty]. \tag{4}$$

It is well known that if u and p solve the system (1)–(2) then the same is true for the rescaled functions $u_\lambda, p_\lambda, \lambda > 0$, defined as

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \qquad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t).$$

The spaces from (4) are called critical since their norms are invariant with regard to the above scaling, that is $\|u\|_{L^\beta(0,T;L^p)} = \|u_\lambda\|_{L^\beta(0,T;L^p)}$ for every $\lambda > 0$. In this sense the Prodi–Serrin conditions are an optimal result.

An analogical situation occurs for ∇u: it was proved in [3] that u with the initial condition from $W^{1,2}_\sigma$ is regular on $(0,T]$ if

$$\nabla u \in L^\beta(0,T;L^p), \qquad 2/\beta + 3/p = 2, \quad p \in (3/2,\infty). \tag{5}$$

This result is also optimal since the spaces from (5) are scale invariant considering ∇u instead of u: $\|\nabla u\|_{L^\beta(0,T;L^p)} = \|\nabla u_\lambda\|_{L^\beta(0,T;L^p)}$. Analogically we could discuss $\nabla^2 u$.

Unlike (4) and (5), in the present paper we are interested in criteria based on only one velocity component. More specifically, we will study criteria based on $u_3, \nabla u_3$ and $\nabla^2 u_3$. We will see that most of these criteria published so far are not optimal in the sense described above. We want to discuss two basic ideas. Firstly, we will use the anisotropic Lebesgue spaces:

Definition 1. Let $\bar{p} = (p_1,p_2,p_3), p_i \in [1,\infty], i = 1,3$. We say that a function f belongs to $L^{\bar{p}}$ if f is measurable on $\mathbb{R}^3$ and the following norm is finite:

$$\|f\|_{L^{\bar{p}}} \equiv \left\| \left\| \|f\|_{L_1^{p_1}} \right\|_{L_2{}^{p_2}} \right\|_{L_3{}^{p_3}} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1,x_2,x_3)|^{p_1} \, dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} dx_3 \right)^{\frac{1}{p_3}}.$$

The anisotropic Lebesgue spaces seem to be convenient for our purposes, since they differentiate between different directions. It can be useful in the situations where additional regularity conditions are imposed only on one velocity component. Namely, in Theorems 2–5 we see, that the use of the anisotropic Lebesgue spaces can improve some results from the literature. Moreover, the result from Theorem 1 formulated in the framework of the anisotropic Lebesgue spaces is almost optimal which is not the case for the corresponding result formulated in the framework of the standard Lebesgue spaces (see the result from [19, Theorem 1], mentioned below).

Secondly, our results and also results from the literature indicate that one is able to prove more easily stronger criteria if conditions are imposed on higher derivatives of the velocity field. It seems natural, since