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The local well-posedness, blow-up criteria and Gevrey regularity of solutions for a two-component high-order Camassa–Holm system

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ABSTRACT

This paper studies the Cauchy problem for a two-component high-order Camassa–Holm system proposed in Escher and Lyons (2015). First, we investigate the local well-posedness of the system in the Besov spaces $B^s_{p,r} \times B^{s-2}_{p,r}$ with $s > \max\{3+\frac{1}{p}, \frac{7}{2}, 4-\frac{1}{p}\}$ and $p, r \in [1, \infty]$. Second, by means of the Littlewood–Paley decomposition technique and the conservative property at hand, we derive a blow-up criteria for the strong solution. Finally, we study the Gevrey regularity and analyticity of the solutions to the system in the Gevrey–Sobolev spaces. In particular, we get a lower bound of the lifespan and the continuity of the data-to-solution mapping.

1. Introduction

In this paper, we consider the following two-component high-order Camassa–Holm system [1]:

$$\begin{cases} m_t = \alpha u_x - a u_x m - u m_x - \kappa \rho \rho_x, \quad m = Au, \\ \rho_t = -u \rho_x - (a-1) u_x \rho, \quad a \in \mathbb{R} \setminus \{1\}, \\ \alpha_t = 0, \end{cases} \tag{1.1}$$

where $Au = (1 - \partial_x^2)^s u$ with $s > 1$, and a, κ are real parameters in $\mathbb{R}$. In [1], by using a geometric approach, the authors showed that the system (1.1) admits a global solution in $C^\infty([0,\infty); C^\infty(\mathbb{S}^1) \oplus C^\infty(\mathbb{S}^1))$ with smooth initial data in $C^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1)$, which implies that the corresponding flow is geodesically complete. Recently, Chen and Zhou [2] investigated the local well-posedness of (1.1) in appropriately chosen Besov

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spaces, and they also establish the persistence properties of the solutions to the system with $s = 1$ in a weighted spaces for a large class of moderate weights.

Noting that for $s = 1$, the system (1.1) reduces to the two-component nonlinear system which models the two-dimensional shallow water waves with constant vorticity [3]:

$$\begin{cases} m_t = \alpha u_x - a u_x m - u m_x - \kappa \rho \rho_x, \\ m = u - u_{xx}, \\ \rho_t = -u\rho_x - (a-1) u_x \rho, \quad a \neq 1. \end{cases} \tag{1.2}$$

The constant α represents the vorticity of the underlying flow and $\kappa > 0$ is an arbitrary real parameter. In [3], the authors not only proved the local well-posedness of the system (1.2) by virtue of a geometric framework, but also studied the blow-up criterion and global strong solutions to the system on the circle. Recently, Guan, He and Yin [4] established the local well-posedness for the Cauchy problem of the system (1.2) in the Besov space $B^s_{p,r} \times B^{s-1}_{p,r}$ with $s > \max\{1 + \frac{1}{p}, 2 - \frac{1}{p}\}$ and $p, r \in [1, \infty]$. They also showed that the solutions to the system (1.2) has exponential decay if the initial data has exponential decay.

If $s = 1$, $\alpha = 0$ and $a = 2$, the system (1.1) becomes the celebrated two-component Camassa–Holm (2CH) system [5]:

$$\begin{cases} m_t + u m_x + 2 u_x m + \kappa \rho \rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases} \tag{1.3}$$

where $m = u - u_{xx}$ and $\kappa = \pm 1$. The Cauchy problem of the 2CH system with $\kappa = -1$ and $\kappa = 1$ have been extensively studied in recent years, see for example [5–9]. For initial datum in the Sobolev spaces and Besov spaces, the local well-posedness for the 2CH system have been investigated in [5,10,9]. The blow-up criteria and global existence of strong solutions to the 2CH system in the Sobolev spaces are established in [10,7,9]. The existence of global weak solutions for the 2CH system with $\kappa = 1$ in the Sobolev spaces was obtained in [8]. Moreover, by virtue of the Cauchy–Kovalevsky theorem, the analyticity of the solution with analytical initial data was investigated in [11,12].

If $s = 1$, $\alpha = 0$ and $\rho \equiv 0$, the system (1.1) reduces to the b-equation:

$$u_t - u_{txx} + (b+1) u u_x = b u u_{xx} + u u_{xxx}, \quad \forall b \in \mathbb{R}. \tag{1.4}$$

The b-equation contains a number of structure phenomena which are shared by solutions of the family of equations [13–15]. The local well-posedness of the b-equation on the line $\mathbb{R}$ and on the circle $\mathbb{S}^1$ are obtained in [13,16]. It is proved that Eq. (1.4) has global solutions [13,16,17]. Moreover, under some sign conditions, the uniqueness and existence of global weak solutions to the b-equation were established in [13,16]. Especially, the b-equation contains two famous shallow water wave equations. One is the completely integrable Camassa–Holm (CH) equation [18]:

$$m_t + u m_x + 2 u_x m = 0, \quad m = u - u_{xx}. \tag{1.5}$$

It has a bi-Hamilton structure and infinite conservation laws in [18,19]. The CH equation has peakons which describe a fundamental characteristic of the traveling waves of largest amplitude, and these solutions can be formulated in the form of $ce^{-|x-ct|}$ with $c > 0$ [20]. The local well-posedness and blow-up phenomena of the CH equation in the Sobolev and the Besov spaces are investigated in [21–26]. For the global existence of the weak and strong solutions, we refer the readers to [21,22,27,24,28]. The global conservative and dissipative solutions of the CH equation were studied in [29–31].

The other one is the Degasperis–Procesi (DP) equation [32]:

$$u_t - u_{xxt} = 4 u u_x - 3 u_x u_{xx} - u u_{xxx}. \tag{1.6}$$