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New results for the Liebau phenomenon via fixed point index

J.A. Cid[a,*], G. Infante[b], M. Tvrdý[c], M. Zima[d]

[a] *Departamento de Matemáticas, Universidade de Vigo, 32004, Pabellón 3, Campus de Ourense, Spain*
[b] *Dipartimento di Matematica e Informatica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy*
[c] *Mathematical Institute, Czech Academy of Sciences, CZ 115 67 Praha 1, Žitná 25, Czech Republic*
[d] *Department of Differential Equations and Statistics, Faculty of Mathematics and Natural Sciences, University of Rzeszów, Pigonia 1, 35-959 Rzeszów, Poland*

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ABSTRACT

We prove new results regarding the existence of positive solutions for a nonlinear periodic boundary value problem related to the Liebau phenomenon. As a consequence we obtain new sufficient conditions for the existence of a pump in a simple model. Our methodology relies on the use of classical fixed point index. Some examples are provided to illustrate our theory. We improve and complement previous results in the literature.

1. Introduction

During the experiments developed in the 1950s, the German cardiologist Gerhart Liebau observed (see [1]) that a periodic compression could produce the circulation of a fluid in a mechanical system without valves to ensure the direction of the flow. This valveless pumping effect is nowadays called the Liebau phenomenon. It was reported to occur for instance in embryonic blood circulation, in applications of nanotechnology and in oceanic currents, see e.g. [2–6] or [7, Chapter 8]. In particular, G. Propst [6] presented an explanation of the pumping effect for flow configurations of several rigid tanks that are connected by rigid pipes. He proved the existence of periodic solutions to the corresponding differential equations for systems of 2 or 3 tanks. However, the apparently simplest configuration consisting of 1 pipe and 1 tank turned out to be, from

* Corresponding author.
E-mail addresses: angelcid@uvigo.es (J.A. Cid), gennaro.infante@unical.it (G. Infante), tvrdy@math.cas.cz (M. Tvrdý), mzima@ur.edu.pl (M. Zima).

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mathematical point of view, the most interesting one, as it leads to the singular periodic problem

$$\begin{cases} u''(t) + a\,u'(t) = \dfrac{1}{u(t)}\left(e(t) - b(u'(t))^2\right) - c, \quad t \in [0,T], \\ u(0) = u(T), \qquad u'(0) = u'(T), \end{cases} \tag{1.1}$$

where u' is the fluid velocity in the pipe (oriented in the direction from the tank to the piston), $T > 0$,

$$a = \frac{r_0}{\rho}, \qquad b = 1 + \frac{\zeta}{2}, \qquad c = \frac{g A_\pi}{A_\tau}, \qquad e(t) = \frac{g\,V_0}{A_\tau} - \frac{p(t)}{\rho},$$

r_0 is the friction coefficient, ρ is the density of the fluid, $\zeta \geq 1$ is the junction coefficient (depending on the particular geometry and smoothness of the junction of the tank and the pipe), g is the acceleration of gravity, A_τ is the cross section of the tank, A_π is the cross section of the pipe (small in comparison with A_τ), V_0 is the total volume (assumed to be constant) of the fluid in the system and p is the T-periodic external force. As a result, from the fluid mechanics point of view, the assumptions

$$a \geq 0, \qquad b > 1, \qquad c > 0, \qquad e \text{ continuous and } T\text{-periodic}$$

are quite natural, from $\zeta \geq 1$ we would even have $b \geq 3/2$. Of course, we are interested in the search of positive solutions of problem (1.1). A detailed justification of the model can be also found e.g. in [7, Chapter 8].

One can observe that if a periodic external force e produces a nonconstant periodic response u then the mean level of the fluid in the tank is higher than the level produced by a constant force with the same mean value. Moreover the increasing of the level is proportional to $\|u'\|^2$.

The change of variables $u = x^\mu$, where $\mu = \frac{1}{b+1}$, was used in [8] in order to overcome the singularity, transforming and simplifying problem (1.1) into the regular BVP

$$\begin{cases} x''(t) + a\,x'(t) = \dfrac{e(t)}{\mu}\,x^{1-2\mu}(t) - \dfrac{c}{\mu}\,x^{1-\mu}(t), \quad t \in [0,T], \\ x(0) = x(T), \qquad x'(0) = x'(T), \end{cases} \tag{1.2}$$

where $0 < \mu < \frac{1}{2}$. By means of the lower and upper solution technique Cid and co-authors [8] provided results on the existence and stability of a positive solution of (1.2).

In our recent paper [9] we considered a generalization of problem (1.2), namely

$$\begin{cases} x''(t) + ax'(t) = r(t)\,x^\alpha(t) - s(t)\,x^\beta(t), \quad t \in [0,T], \\ x(0) = x(T), \qquad x'(0) = x'(T), \end{cases} \tag{1.3}$$

under the assumption

$$a \geq 0, \qquad r, s : [0,T] \to \mathbb{R} \text{ are continuous and } 0 < \alpha < \beta < 1. \tag{H0}$$

Of course, to extend the obtained solution of the boundary value problem (1.3) (on $[0,T]$) to a T-periodic solution of the corresponding differential equation, we would have to assume that r and s are also T-periodic. Making use of a shifting argument and Krasnosel'skiĭ's expansion/compression fixed point theorem on cone, we succeeded in [9] to improve the existence results from [8].

Furthermore, Torres in [7, Chapter 8] obtained a priori bounds for the periodic solutions of (1.1), which together with the Brouwer degree theoretical arguments, led to an alternative existence result.

We point out that the assumption

$$\min_{t \in [0,T]} e(t) > 0$$

is a common feature of the existence results in [7–9]. The goal of this paper is twofold: first, to improve the main results from [9] and, second, to obtain explicit sufficient conditions for the existence of periodic