Nonlinear Analysis: Real World Applications 35 (2017) 470–482

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Large solutions for non-divergence structure equations with singular lower order terms

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ARTICLE INFO

Article history:
Received 14 September 2016
Accepted 1 December 2016

Keywords:
Semi-linear elliptic equations
Large solutions
Boundary behavior
Uniqueness

ABSTRACT

In this paper we investigate infinite boundary value problems associated with the semi-linear PDE $Lu = k(x)f(u)$ on a bounded smooth domain $\Omega \subset \mathbb{R}^n$, where L is a non-divergence structure, uniformly elliptic operator with singular lower order terms. The weight k is a continuous non-negative function and f is a continuous nondecreasing function that satisfies the Keller–Osserman condition. We study a sufficient condition on k that ensures existence of a large solution u. In case the lower order terms of L are bounded, under further assumptions on f and k we establish asymptotic bounds of solutions u near the boundary $\partial\Omega$ and, as a consequence a uniqueness result.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary. We consider a non-divergence uniformly elliptic second order differential operator

$$Lu = a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + c(x)u.$$

Here and throughout the paper, the summation convention over repeated indices from 1 to n is in effect. In this paper we will assume that L is uniformly elliptic on Ω. More specifically, we require that $[a_{ij}(x)]$ is a $n \times n$ symmetric matrix of continuous functions on Ω satisfying the following ellipticity condition

(C-1): There are constants $0 < \lambda \leq \Lambda$ such that

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^n.$$

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http://dx.doi.org/10.1016/j.nonrwa.2016.12.002

We also have occasion to assume that the lower order term coefficients are locally bounded Borel functions such that

(C-2): $d(x)\sum |b_i(x)| \to 0$ as $d(x) \to 0$.

(C-3): $c(x) \le 0$ and $d^2(x)|c(x)| \le \eta(d(x))$ for all $x \in \Omega$ and $d(x) \le \delta_0$. Here δ_0 is a positive constant and $\eta : (0, \delta_0] \mapsto (0, \eta(\delta_0)]$ is an increasing function that satisfies the Dini condition

$$\int_0^{\delta_0} \frac{\eta(s)}{s} ds < \infty. \tag{1.1}$$

In (C-2), (C-3) and throughout this paper $d(x)$ represents the distance from x to the boundary $\partial\Omega$. Sometimes we write $L_0 u = a_{ij}(x)u_{x_i x_j}$.

In Section 3 we study the following infinite boundary value problem

$$Lu = k(x)f(u) \quad \text{in } \Omega,\ u \to \infty \text{ as } \ x \to \partial\Omega, \tag{1.2}$$

where f is a nondecreasing function satisfying appropriate growth conditions at infinity, and k is a C_Ω-positive function which may be unbounded in Ω. Recall that a function $\kappa : \Omega \mapsto [0, \infty)$ is C_Ω-positive if $\kappa(x_0) = 0$ for some $x_0 \in \Omega$ implies there is $O \subset\subset \Omega$ such that $x_0 \in O$ and $\kappa > 0$ on ∂O. This concept was introduced by A. Lair in [1]. We also require that k satisfies the following condition

(k-1): $d^2(x)k(x) \le \eta(d(x))$ for all $x \in \Omega$ with $d(x) \le \delta_0$, where δ_0 and η are as in (C-3).

In the special case $L = \Delta$, the problem of existence and uniqueness of solutions to Problem (1.2) was investigated by several authors under different assumptions on the terms $k(x)$ and $f(t)$. In this particular case, we refer to [2–11] and the references therein for work related to existence and uniqueness of solutions to (1.2). The case $L = L_0$ was studied by L. Veron in the paper [12] when the right-hand side term has the form $k(x)t^q$ under the assumption that $0 < c \le k(x) \le C$ on Ω for some constants c and C and $1 < q < \infty$. When the coefficients a_{ij}, b_i of L are in $C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$ and $c(x) \equiv 0$ in Ω, we refer to the works of Bandle and Marcus in [13] for their work on existence and uniqueness of solutions to (1.2).

The contributions in this paper are two fold: We treat the case when the lower order terms of L are singular. Moreover, as far as the existence of solutions to (1.2) with singular weights h is concerned, our conditions on the nonlinearity f are, to the best of our knowledge, weaker than those studied in the literature.

The paper is organized as follows. In Section 2, we state some preliminary results and conditions on the nonlinearity f that will be needed later. Existence of solutions to (1.2) is stated and proved in Section 3. In Section 4, under additional conditions on f and on k we obtain asymptotic boundary estimates of solutions u of (1.2) near the boundary $\partial\Omega$. Here we require the coefficients of L to be bounded in Ω. Finally, we state and prove a uniqueness result for solutions to (1.2) under the additional assumption that f is convex. Here we employ the method introduced by Marcus and Veron in their paper [14]. See also [4,5].

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^2 boundary $\partial\Omega$. Throughout the paper, it will be convenient to use the following notations. Given $\delta > 0$,

$$\Omega_\delta := \{x \in \Omega : d(x) < \delta\} \quad \text{and} \quad \Omega^\delta := \{x \in \Omega : d(x) > \delta\}.$$

Since Ω is a bounded C^2 domain, there is $\mu > 0$ such that $d \in C^2(\Omega_\mu)$ and $|\nabla d| = 1$ in Ω_μ (see [15], Lemma 14.16 for a proof).

By modifying the distance function d appropriately we can suppose that d is a positive C^2 function on Ω. For instance one can use $(1-\psi)d + \psi$ instead of d where $\psi \in C_c^2(\Omega)$ is a cut-off function with $0 \le \psi \le 1$