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Semilinear subdiffusion with memory in the one-dimensional case

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A B S T R A C T

For $\nu \in (0,1)$, we consider the semilinear integro-differential equation on the one-dimensional domain $\Omega = (a,b)$ in the unknown $u = u(x,t)$

$$\mathbf{D}_t^\nu u - \mathcal{L}_1 u - \int_0^t \mathcal{K}(t-s)\mathcal{L}_2 u(\cdot,s)ds = f(x,t,u) + g(x,t)$$

where $\mathbf{D}_t^\nu$ is the Caputo fractional derivative and $\mathcal{L}_1$ and $\mathcal{L}_2$ are uniform elliptic operators with time-dependent smooth coefficients. Under certain structural conditions on the nonlinearity f, the global existence and uniqueness of classical solutions to the related initial–boundary value problems are established, via the so-called continuation argument approach. The key point is looking for suitable *a priori* estimates of the solution in the fractional Hölder spaces.

1. Introduction

Let $\Omega = (a,b) \subset \mathbb{R}$ be a segment, with a boundary $\partial\Omega = \{a\} \cup \{b\}$. For an arbitrary fixed time $T > 0$, we denote

$$\Omega_T = \Omega \times (0,T) \qquad \text{and} \qquad \partial\Omega_T = \partial\Omega \times [0,T].$$

For a fixed $\nu \in (0,1)$, we consider the semilinear equation in the unknown function $u = u(x,t) : \Omega_T \to \mathbb{R}$

$$\mathbf{D}_t^\nu u - \mathcal{L}_1 u - \mathcal{K} \star \mathcal{L}_2 u = f(x,t,u) + g(x,t), \tag{1.1}$$

subject either to the Dirichlet boundary condition (DBC)

$$u(x,t) = \psi(x,t) \quad \text{on } \partial\Omega_T, \tag{1.2}$$

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or to the Neumann boundary condition (NBC)

$$c_1(x,t)\frac{\partial u}{\partial x}(x,t) + \left(\mathcal{K} \star d_1 \frac{\partial u}{\partial x}\right)(x,t) = \psi_1(x,t) \quad \text{on } \partial\Omega_T, \tag{1.3}$$

where the functions ψ, ψ_1, c_1, d_1 are prescribed. The equation is supplemented with the initial condition

$$u(x,0) = u_0(x) \quad \text{in } \bar{\Omega}, \tag{1.4}$$

for some given initial datum u_0.

Here, the *star* denotes the usual time-convolution product on $(0,t)$, namely,

$$(h_1 \star h_2)(t) = \int_0^t h_1(t-s)h_2(s)ds,$$

while the symbol $\mathbf{D}_t^\nu$ stands for the Caputo fractional derivative of order ν with respect to t (see e.g. (2.4.1) in [20]), defined as

$$\mathbf{D}_t^\nu u(x,t) = \frac{1}{\Gamma(1-\nu)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\tau) - u(x,0)}{(t-\tau)^\nu} d\tau,$$

Γ being the Euler Gamma-function. An equivalent definition reads

$$\mathbf{D}_t^\nu u(x,t) = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{1}{(t-s)^\nu} \frac{\partial u}{\partial s}(x,s)ds.$$

In the limit cases $\nu = 0$ and $\nu = 1$, the Caputo fractional derivatives of $u(x,t)$ boil down to $u(x,t)$ and $\frac{\partial}{\partial t}u(x,t)$, respectively.

The terms $f(x,t,u)$ and $g(x,t)$ represent external sources, the first depending (possibly in a nonlinear way) on the variable u itself.

Finally, coming to the operators involved, $\mathcal{L}_i$ are linear elliptic operators of the second order with time-dependent coefficients, namely,

$$\mathcal{L}_1 u = a_2 \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial u}{\partial x} + a_0 u,$$
$$\mathcal{L}_2 u = b_2 \frac{\partial^2 u}{\partial x^2} + b_1 \frac{\partial u}{\partial x} + b_0 u,$$

where $a_i = a_i(x,t)$ and $b_i = b_i(x,t)$.

The motivations in the study of problems like (1.1)–(1.4) arise from a variety of investigations in several fields, such as Biorheology, Geophysics, Chemistry and Physics (see, e.g., [4,6,7,11,16,26,27,30,31,35,39]). With reference to porous media with memory [7,8,19], we can interpret the function u as the fluid mass per unit volume. Letting v be the flux rate and p be pressure of the fluid, following [7] we adopt for any $(x,\tau) \in \Omega_T$ the constitutive assumptions

$$v(x,\tau) = -\left(\gamma_1 + \gamma_2 \mathbf{D}_\tau^\theta\right) \frac{\partial p}{\partial x}(x,\tau), \tag{1.5}$$

$$p(x,\tau) = \gamma_3 u(x,\tau), \tag{1.6}$$

$$\frac{\partial u}{\partial \tau}(x,\tau) + \frac{\partial v}{\partial x}(x,\tau) = q(u(x,\tau)), \tag{1.7}$$

where $q(u)$ represents a source term, $\theta \in (0,1)$, and $\gamma_1 \neq 0$, $\gamma_2 \neq 0$ and γ_3 are given constants. The values θ, γ_1 and γ_2 are so called "memory parameters". Note that if $\gamma_2 = 0$, relation (1.5) collapses into the classical Darcy law, which describes fluid through a porous medium. However, the passage of the fluid through the porous matrix may cause a local variation of the permeability. For instance, the flow may perturb the porous